Correction to the Earth's orbit due to the Sun-Earth-Moon interaction

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Abstract

In this article we studied the influence of the motion of the Moon on the Earth orbit around the Sun. We started by considering the exact Newtonian gravitational interaction in the Sun-Earth-Moon system and, using the following mass hierarchy: $M_\odot \gg M_{Earth} \gg M_{Moon}$, derived the corresponding correction to the "naive" Earth-Sun potential energy. Without the obtained correction (and neglecting the other effects) the Earth's orbit is well known and corresponds to the Kepler's problem, with the obtained correction the Earth's orbit deviates from a perfect ellipse, for example, the Earth's perigee starts to shift by a small angle every year. In this study, this shift was calculated to be about 7.9 arcseconds per century and was compared to the other known effects, which lead to the Earth perigee precession. We also estimated similar interaction for Jupiter and its Galilean moons.

1 Introduction

The motions of bodies in our solar system are known to be caused by the gravitational interaction between the planet, planet's moons and the Sun, which gives us the shape of the planet's orbit. In this work, we focus on the shape of Earth's orbit which is said to be a perfect ellipse, if there is only gravitation interaction between the Earth and Sun. However, there are additional interactions such as, impact of other planets, which cause the orbit to deviate from an ellipse. In our case, when the Moon orbits around Earth it introduces an identical correction. Particularly, when it is closer to the

Sun it has a stronger force on it and when it is farther from the Sun the force weakens. This outcome causes further precession of the Earth's orbit and this what will be examined in this project. This study is significant because it can help us determine the path of Earth and whether it has any implication to our climate, in the future. Alternatively, calculating accurate data can benefit us to study Earth-like planets in other systems and their motions around their parent star.

2 Theory outlines

If there is only Newtonian gravitational interaction between the Earth and Sun, then it is known that the Earth's orbit is a perfect ellipse as seen in Fig. 1, the left panel. However, there are additional interactions which make the orbit deviate from an ellipse. For example, general relativity corrections, relativistic corrections, influence of other planets, and so on. When the Moon orbits around Earth it will introduce a similar correction as well: when it is closer to the Sun it has a stronger force on it and when it is farther from the Sun the force weakens. This effect causes an additional precession of the Earth's orbit, as shown in Fig. 1, the right panel, and this is what we study in this research.

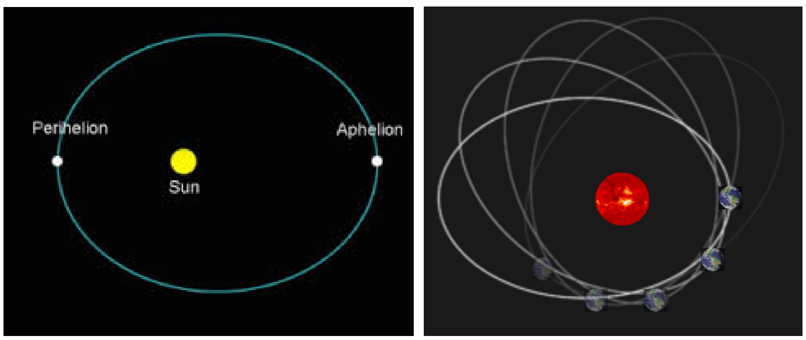

Figure 1: Earth's elliptical orbit (on the left) vs. the precession of the Earth's orbit due to Sun-Moon interaction (on the right).

2.1 The interaction of the Sun-Earth-Moon system

In order to calculate the deviation in the Earth's orbit from an ellipse, we write the exact potential energy of the interaction between the Sun, Moon and Earth

$$U = -G\frac{M_\odot M_M}{r_1} - G\frac{M_\odot M_E}{r_2} - G\frac{M_E M_M}{r}, \tag{1}$$

where U is the potential energy, $G = 6.67 \times 10^{-11}\, Nm^2/kg^2$ is the fundamental gravitational constant, $M_\odot$ is the mass of the Sun, M_M is the mass of the Moon, M_E is the mass of Earth, r_1 is the distance between the Sun and Moon, r_2 is the distance between the Sun and Earth, and r is the distance between the Earth and Moon, as shown in Fig. 2.

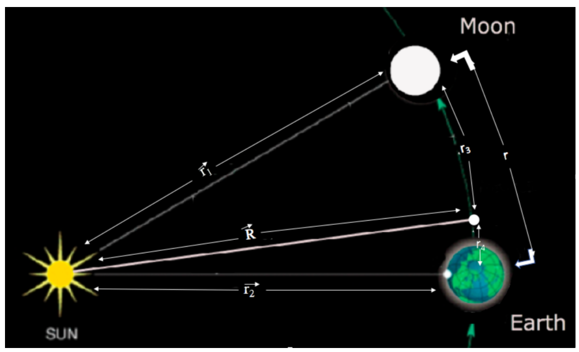

Figure 2: Sun-Earth-Moon System

Let us introduce the position of the Earth-Moon center of mass calculated relative to the Sun, $\vec{R}$, and present the positions of the Moon and Earth as

$$\begin{cases} \vec{r}_1 = \vec{R} + \vec{r}_3 \\ \vec{r}_2 = \vec{R} + \vec{r}_4 \end{cases}, \tag{2}$$

here we introduced the relative positions of the Moon and Earth $\vec{r}_3$, $\vec{r}_4$ correspondingly. Due to the mass differences it is important to recognize the following hierarchy of distances and masses

$$\begin{array}{l} M_M \ll M_E \ll M_\odot \\ r_4 \ll r_3 \ll r_2 \approx r_1 \approx R. \end{array} . \tag{3}$$

We can simplify the exact potential energy (1) by using the differences in the distances Eq.(3), and for the Sun-Moon interaction we can write

$$U_{S-M} = -G\frac{M_\odot M_M}{r_1} = -G\frac{M_\odot M_M}{R}\frac{1}{\sqrt{1 + \frac{2(\vec{r}_3\cdot\vec{R})}{R^2} + \frac{r_3^2}{R^2}}}, \tag{4}$$

where we used Eq.(2) for the Sun-Moon distance r_1. It is important to note that the ratio $r_3/R \sim 10^{-3}$ is small and has a small effect on the Sun-Moon interaction, using Taylor series and keeping only the first two orders with respect to r_3/R, one can obtain

$$U_{S-M} \approx -G\frac{M_\odot M_M}{R}\left(1 - \frac{(\vec{R}\cdot\vec{r_3})}{R^2} - \frac{r_3^2}{2R^2} + \frac{3(\vec{R}\cdot\vec{r_3})^2}{2R^4}\right), \tag{5}$$

here we have three different terms: the monopole term, which corresponds to the pure Kepler's problem; the first order correction - the dipole term, which is proportional to r_3/R; and the second order term - the quarupole term, which is proportional to $(r_3/R)^2$. We will later see that the dipole terms cancel out and only the quadrupole terms survive. Similarly, the Sun-Earth interaction can be written as

$$U_{S-E} \approx -G\frac{M_\odot M_E}{R}\left(1 - \frac{(\vec{R}\cdot\vec{r_4})}{R^2} - \frac{r_4^2}{2R^2} + \frac{3(\vec{R}\cdot\vec{r_4})^2}{2R^4}\right). \tag{6}$$

Let us add the interactions (5) and (6) together and write the total interaction between the Sun and the Earth-Moon system as

$$U_{S-E} + U_{S-M} = -G\frac{M_\odot(M_E + M_M)}{R} + \delta U_1 + \delta U_2, \tag{7}$$

where δU_1 and δU_2 contain the first and second order corrections respectively. It can be shown that δU_1 cancels out, as seen in Eq (8).

$$\delta U_1 = \frac{GM_\odot}{R^3}[M_E(\vec{R}\cdot\vec{r_4}) + M_M(\vec{R}\cdot\vec{r_3})] = 0, \tag{8}$$

this is because $\vec{r_3}$ and $\vec{r_4}$ are calculated relative to the Earth-Moon center of mass.

$$M_E\vec{r_4} + M_M\vec{r_3} = 0. \tag{9}$$

For the quadrupole term δU_2, we can neglect the Earth contribution since $M_E r_4^2 \ll M_M r_3^2$, where factors of substitution were used in deriving the equation such as $\vec{r_3} \approx \vec{r}$ and $\vec{r_4} = \vec{r_3}(\frac{M_M}{M_E})$, where mass of Moon and Earth was calculated, to be approximately 0.01. So we finally get

$$\delta U_2 = G\frac{M_\odot M_M}{2R^3}\left(r^2 - 3\frac{(\vec{r}\cdot\vec{R})^2}{R^2}\right), \tag{10}$$

and our potential energy of interaction for Sun-Moon and Earth system (1), with the obtained correction is

$$U = -G\frac{M_\odot(M_E + M_M)}{R} + G\frac{M_\odot M_M}{2R^3}\left(r^2 - 3\frac{(\vec{r}\cdot\vec{R})^2}{R^2}\right). \tag{11}$$

To simplify further, we average the quadrupole term (10) over Moon's rotation around the Earth

$$\langle r^2 - 3\frac{(\vec{r}\cdot\vec{R})^2}{R^2}\rangle = \langle r^2 - 3r^2\cos^2\theta\rangle = -\frac{1}{2}r^2, \tag{12}$$

here θ is the angle between the position of the Earth-Moon center of mass and the Moon and we used

$$\langle\cos^2\theta\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\, d\theta = 1/2. \tag{13}$$

So our effective interaction becomes

$$U = -G\frac{M_\odot(M_M + M_E)}{R} - \frac{GM_\odot M_M r^2}{4R^3}, \tag{14}$$

where the first term is the well known Kepler problem with known solution and the second term is

$$\delta U = -\frac{\beta}{R^3}, \tag{15}$$

here $\beta = GM_\odot M_M r^2/4$. Equation (15) represents the correction to the Kepler problem and will result in the change of the Earth's orbit.

2.2 The shift of the Earth's perigee

Now let us calculate the shift of Earth perigee due to the correction Eq. (15). The change in the orbital angle for one complete orbit is

$$\Delta\theta = \oint d\theta = 2\int_{r_p}^{r_a}\frac{d\theta}{dr}dr, \tag{16}$$

where the integral is taken over one complete orbit, r_p is perigee and r_a is the apogee of the Earth. The change of the angle over the change of the radius $d\theta/dr$ as a function of r can be presented as

$$\frac{d\theta}{dr} = \frac{\dot\theta}{\dot r} = \frac{L}{mr^2}\frac{1}{\sqrt{\frac{2}{m}(E - U_{eff}(r))}}, \tag{17}$$

here we expressed the angular velocity $\dot{\theta}$ through the orbital angular momentum L using

$$L = mr^2\dot{\theta}, \tag{18}$$

L is the angular momentum, $m = M_E + M_M$, and r is the distance between the Earth and Sun. For the radial velocity $\dot{r}$, we used the conservation of mechanical energy for radial motion, see Ref. [1],

$$E = \frac{m\dot{r}^2}{2} + U_{eff}(r), \tag{19}$$

where the effective potential

$$U_{eff}(r) = \frac{L^2}{2mr^2} - \frac{\alpha}{r}. \tag{20}$$

where $\alpha = GM_\odot m$. Finally, the integral from Eq.(16) turns into

$$\Delta\theta = 2\int_{r_1}^{r_2} \frac{L}{mr^2} \frac{dr}{\sqrt{\frac{2}{m}(E - U_{eff}(r))}} \tag{21}$$

This integral was calculated after multiple substitutions with lengthy and sophisticated computations with the result being

$$\Delta\theta = 2\pi + \left(\frac{6\pi m^2 GM_\odot M_M r^2 \alpha}{L^4}\right), \tag{22}$$

where 2π corresponds to perfect ellipse of the Earth orbit around the Sun.

3 Result

Now let us simplify Eq(20) by substituting $m = M_E$, $\alpha = GM_\odot M_E$ and $L^4 = {M_E}^4 G{M_\odot}^2 R^2$, since $L = mr^2\omega$ and $\omega = \sqrt{GM \odot / r^3}$, and this gives us

$$\Delta\theta = 2\pi + \frac{3\pi}{2}\left(\frac{r}{R}\right)^2\left(\frac{M_M}{M_E}\right), \tag{23}$$

where r is the orbital radius of the Moon and R is the distance of Earth from the Sun. After substituting M_M, M_E, r, R with their actual numbers and calculating the angle, the result is

$$\delta\theta = \Delta\theta - 2\pi = 3.83 * 10^{-7} rads = 2.2 * 10^{-5} degs = 7.9 * 10^{-2} arcseconds. \tag{24}$$

So in one year the Earth's orbit would shift by

$$\delta\theta = 7.9 * 10^{-2} arc - seconds \tag{25}$$

and by 7.9 arc-seconds per century. Comparing this precession to the precession due to Jupiter and Venus we get 1158 arc-seconds per century. Additionally, calculation according to general relativity gives us 3.8 arc-seconds per century, see Ref.[2] and if done relativistically the correction is one arc-second per century. Using Eq(23), the precession of Jupiter caused by its four Galilean moon's was also calculated and it is estimated to be $4.9 * 10^{-4}$ arcseconds every 12 Earth years or 1 year of Jupiter.

4 Conclusion

In this project, the motion of Earth's orbit around the Sun was studied, and described to be a perfect ellipse, if there was only Earth and Sun gravitation interaction, also known as the Kepler problem. However, the Moon has additional interaction with the Earth-Sun system, which introduces correction to the elliptical orbit of Earth. This correction was calculated by introducing exact gravitational interaction in the Sun-Earth-Moon system and, using the following mass hierarchy: $M_\odot \gg M_{Earth} \gg M_{Moon}$, the correction to the "simple" Earth-Sun potential energy was derived. Later, this derived correction was used to calculate the precession of Earth perigee, which is calculated to be 7.9 arc-seconds per century. Although, the shift is not large it is comparable to the precession due to general relativity, relativistic correction and to Jupiter and Venus. Similarly, the precession of Jupiter due to its four Galilean moons was estimated as well, with its shift being $4.9 * 10^{-4}$ arcseconds every 12 Earth years. Further research can include in calculating the precession of Moon's orbit due to Earth-Moon interaction with the Sun.

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[2] Abhijit Biswas and Krishnan R. S. Mani, *Relativistic Perihelion Precession of Orbits of Venus and the Earth*, https://arxiv.org/abs/0802.0176