

Science Technology Engineering Math

STUDY GUIDE

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Preface

Drs. Zahidur Rahman, Yves Ngabonziza, Paul West, and Tao Chen are professors in the Mathematics, Engineering and Computer Science Department, and Drs. Dionne Miller and Xin Gao are professors in the Natural Science Department.

The authors collaboratively prepared this guide for STEM students. It has some of the following features:

- This guide is designed to be a reference study material for STEM courses such as physics, chemistry, mathematics, electrical, civil, and mechanical engineering.
- It can be used as a reference book for most of the important formulas of the afore-mentioned topics, with useful examples to help students master their class work, exams and projects.
- This guide is organized to emphasize the main topics taught in these courses.
- Students are strongly advised to follow their instructors' recommendations when preparing for any exam, project and homework, but this study guide will be a useful tool.
- This study guide is intended to be used as a supplement to the assigned textbook and lecture notes.

The authors hope that the users of this guide find it informative and useful. If you have any suggestions, comments, or any mistakes please email: zrahman@lagcc.cuny.edu.

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PART I NEWTONIAN MECHANICS

Motion

Quantities Used to Describe Motion (Definitions and Notations):

Time ($t = \Delta t = t_2 - t_1$): In this part of Physics, we take time as a time interval, which is the difference between two moments (t_2 and t_1)

Distance (d): the length of the path traveled by an object (scalar)

Displacement ($\vec{d} = \vec{x}_2 - \vec{x}_1$): the change in position (vector)

Speed: the ratio between distance and time (scalar)

Velocity (\vec{v}): the time rate of change in position, so it is defined in a formula as follows:

$$\vec{v} = \frac{\vec{d}}{t} = \frac{\vec{x}_2 - \vec{x}_1}{t}$$

Acceleration (\vec{a}): the time rate of change in velocity, so it is defined in a formula as follows:

$$\vec{a} = \frac{\Delta \vec{v}}{t} = \frac{\vec{v}_2 - \vec{v}_1}{t}$$

Synthesis Example:

The position function of an object is given as: $x = 2t^3 - 3t^2$, where x is in meters and t in seconds. (a) What is the velocity of the object at $t = 2s$, (b) What is the acceleration of the object at $t=2s$, (c) At what time does the object reach its minimum position from $t=0$ to $t=5s$?

Solution:

$$(a) v(t = 2) = \frac{dx}{dt} = 6t^2 - 6t = 6 \times 2^2 - 6 \times 2 = 12 \text{ m/s}$$

$$(b) a(t = 2) = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = 12t - 6 = 12 \times 2 - 6 = 18 \text{ m/s}^2$$

(c) To find the minimum position (x), we take the first derivative of x and let it be 0, i.e. $6t^2 - 6t = 0$, we get $t=0$ or $1s$ as two times for two extreme values. Comparing the positions at these two times, we get the position at $t=1s$ is minimum, which is $x(t = 1) = -1m$

Different types of Motion

Uniform Motion: can also be called “constant velocity motion”. Since velocity is a vector, constant velocity means both direction and magnitude of the velocity is constant. In other words, an object doing uniform motion would be moving with constant speed in a given direction. In this case, speed is just the magnitude of the velocity, so the following formula can be used in this kind of motion:

$$\vec{v} = \frac{\vec{d}}{t},$$

$$\vec{a} = 0.$$

Constant acceleration motion in a Straight Line: If an object is doing accelerating motion, and the acceleration is constant, we call it “Constant Acceleration Motion”. It means the following:

\vec{v} is not constant, but \vec{a} is constant. The following formulas can be derived from the basic definitions of velocity and acceleration listed in part 1.1.1:

$$\vec{v}_f = \vec{v}_i + \vec{a}t$$

$$\vec{d} = \vec{x}_f - \vec{x}_i = \vec{v}_i t + \frac{1}{2} \vec{a}t^2$$

$$2\vec{a} \cdot \vec{d} = v_f^2 - v_i^2$$

$$\vec{d} = \vec{x}_f - \vec{x}_i = \frac{\vec{v}_i + \vec{v}_f}{2} t$$

Synthesis Example:

A car moving along a straight line with constant acceleration covered 80m in 5s. At $t=5s$, the velocity is 20m/s, what was the initial velocity and the acceleration?

Solution:

Based on what is given, and using the formulas for Constant Acceleration Motion, we have the following equations:

$$\begin{cases} 80 = v_i \times 5 + \frac{1}{2} \times a \times 5^2 \\ 20 = v_i + a \times 5 \end{cases}$$

NOTE: We assign positive values to all vectors in the formulas because all the vectors are in the same direction.

Solve this group of equations, we get:

$$\begin{cases} a = 1.6 \text{ m/s}^2 \\ v_i = 12 \text{ m/s} \end{cases}$$

Free Fall: the motion an object would do under the influence of only one force –Gravity. So if gravity is the only force acting on the object, the acceleration of the object will be gravitational acceleration, “g”, which is constant with a value of 9.8 m/s^2 . Obviously, free fall is a perfect example of Constant Acceleration Motion. And it makes a special case of

Constant Acceleration Motion. The following formulas can be used for this kind of motion:

$$\begin{aligned}\vec{v}_f &= \vec{v}_{iy} + \vec{g}t \\ \vec{h} = \vec{x}_f - \vec{x}_i &= \vec{v}_{iy}t + \frac{1}{2}\vec{g}t^2 \\ 2\vec{g} \cdot \vec{h} &= v_{2y}^2 - v_{1y}^2 \\ \vec{h} = \vec{x}_f - \vec{x}_i &= \frac{\vec{v}_{iy} + \vec{v}_{fy}}{2}t\end{aligned}$$

As indicated, we just need to change \vec{a} to \vec{g} .

An even more special case is that when an object doing free fall motion is dropped from rest, the initial velocity will be 0: $\vec{v}_i = 0$ and the above formulas can be changed to even simpler forms:

$$\begin{aligned}\vec{v} &= \vec{g}t \\ \vec{h} = \vec{x}_2 - \vec{x}_1 &= \frac{1}{2}\vec{g}t^2 \\ 2\vec{g} \cdot \vec{h} &= v_2^2 \\ \vec{h} = \vec{x}_2 - \vec{x}_1 &= \frac{\vec{v}}{2}t\end{aligned}$$

Synthesis Example:

A ball is thrown vertically upward from ground level. It takes the stone 5s to reach the highest position, (a) What was the initial velocity when the stone was thrown from the ground? (b) What was the maximum height the stone reached?

Solution:

NOTE:

(1) The initial velocity is upward, but the gravitational acceleration is downward, so we need to pay extra attention to the signs we assign to the vectors in the above formulas for free fall. To do that, we **choose** upward as positive direction, then downward is negative automatically.

(2) Implied information of this problem is that at the highest position, the vertical velocity is 0. So we have the following information:

(a)

$$0 = v_i - g \times 5 = v_i - 9.8 \times 5$$

so: $v_i = 49 \text{ m/s}$, upward

$$(b) h = v_i \times 5 - \frac{1}{2} \times 9.8 \times 5^2 = 49 \times 5 - \frac{1}{2} \times 9.8 \times 5^2 = 122.5 \text{ m}$$

Projectile Motion: a 2-D motion which can be decomposed into 2 separate motions in 2 perpendicular directions: horizontal (x) and vertical (y) directions.

Horizontally (x-direction): uniform motion

Vertically (y-direction): free fall motion

So:

In the x-direction: $d_x = v_{ix}t$

In the y-direction:

$$\vec{v}_{fy} = \vec{v}_{iy} + \vec{g}t$$

$$\vec{h} = \vec{x}_f - \vec{x}_i = \vec{v}_{iy}t + \frac{1}{2}\vec{g}t^2$$

$$2\vec{g} \cdot \vec{h} = v_{2y}^2 - v_{1y}^2$$

$$\vec{h} = \vec{x}_f - \vec{x}_i = \vec{v}_{ty}t - \frac{1}{2}\vec{g}t^2$$

$$\vec{h} = \vec{x}_f - \vec{x}_i = \frac{\vec{v}_{iy} + \vec{v}_{fy}}{2}t$$

The connection between the motion in the x-direction and that in the y-direction is the flying time, t. For example:

Synthesis Example:

A ball is kicked from the ground with an initial speed 30m/s and an angle $\theta_0 = 30^\circ$ with the horizontal direction. (a) what is the maximum height the ball reaches in the vertical direction? (b) what is the distance traveled horizontally by the ball?

Solution:

$$v_{ix} = v_i \cos 30^\circ = 30 \times \cos 30^\circ \approx 26 \text{ m/s}$$

$$v_{iy} = v_i \sin 30^\circ = 30 \times \sin 30^\circ = 15 \text{ m/s}$$

Using the implied information that the vertical velocity at the highest position is 0 and assuming upward as positive direction, we get the following equation:

$$\begin{aligned} 0 &= 15 - 9.8t \\ t &= 1.53 \text{ s} \end{aligned}$$

Considering the fact that from the ground to the highest point and from the highest point to the ground are two symmetrical processes and they take same amount of time, $t = 1.53 \text{ s}$.

Then the total flying time of the ball is: $2t = 2 \times 1.53 = 3.06 \text{ s}$

Then the total horizontal distance traveled by the ball is:

$$d_x = v_{ix}t = 26 \times 3.06 = 79.56 \text{ m}$$

Newton's Laws: Explaining Motion

Newton's 1st Law (a special case of Newton's 2nd Law): A body at rest stays at rest and a body in motion stays in motion unless an external force is exerted on the body. If there is no net force on a body, the body remains at rest if it is initially at rest or moves in a straight line at constant speed if it is in motion. In formulas, it can be expressed as follows:

If $\sum \vec{F}_{\text{ext}}=0$, $\vec{a}=0$.

Moreover, if $\vec{a}=0$, $\sum \vec{F}_{\text{ext}}=0$.

Synthesis Example:

A crate is being pushed by a 50N force to move to the right. The motion is along a straight line with a constant speed 3m/s . What is the frictional force exerted on the crate in this process?

Solution:

Since the crate is doing uniform motion (constant velocity), acceleration $a=0$. According to Newton's 1st Law, if $\vec{a} = 0$, $\sum \vec{F} = 0$. In the horizontal direction, there are only two forces acting on the crate: the exerted 50N force and the unknown frictional force. These two must add up to be 0. So the frictional force is 50N to the left. In formulas:

$$\sum \vec{F} = \vec{F}_{\text{ex}} + \vec{F}_{\text{fr}}$$

\vec{F}_{ex} and \vec{F}_{fr} are in opposite directions, so we assign opposite signs to them. If only considering the magnitude and choose "right" as positive direction, the above equation becomes:

$$\sum F = F_{\text{ex}} - F_{\text{fr}} = 50 - F_{\text{fr}}=0$$

so $F_{\text{fr}} = 50 \text{ N}$, to the left.

Newton's 2nd Law: The net force on a body with mass m is related to the body's acceleration \vec{a} by

$$\sum \vec{F}_{\text{ext}}=m \vec{a}$$

From here, we can easily see that Newton's 1st Law is just a special case of Newton's 2nd Law when $\sum \vec{F}_{\text{ext}}=0$.

It can also be expressed in following forms:

$$\vec{a}=\frac{\sum \vec{F}_{\text{ext}}}{m}$$

$$m = \frac{|\sum \vec{F}_{\text{ext}}|}{|\vec{a}|}$$

Newton's 3rd Law: If a force \vec{F}_{BC} acts on body B due to body C, then there is a force \vec{F}_{CB} on body C due to body B:

$$\vec{F}_{BC} = -\vec{F}_{CB}$$

It means that these two forces have equal magnitude and opposite directions.

NOTE: In understanding the action and reaction forces, one has to realize that they do have same size and opposite in direction. Mathematically, they seem to be able to cancel each other. However, they cannot be canceled because they are acted on two different bodies.

Circular Motion

Centripetal Acceleration: $a_c = \frac{v^2}{r}$, always directing to the center

Centripetal Force: According to Newton's 2nd Law: $\vec{F}_c = m\vec{a}_c$, then $F_c = \frac{mv^2}{r}$.

Universal Gravitation: $F = \frac{GM_1M_2}{R^2}$

Synthesis Example:

A Ferris wheel has a radius of 15m and turns at a speed of 9 m/s. What is the normal force acting on a rider whose mass is 50kg at the bottom of the Ferris wheel?

Solution:

$$a_c = \frac{v^2}{r} = \frac{9^2}{15} = 5.4 \text{ m/s}^2$$

$$F_c = \frac{mv^2}{r} = \frac{50 \times 9^2}{15} = 270 \text{ N}$$

At the bottom of the Ferris wheel, there are two forces acting on the rider in the vertical direction: weight (W: downward) and the normal force (N: upward). We also know that the centripetal force at the bottom is pointing upward. So:

$$N - W = F_c = 270 \text{ N}$$

and
so:

$$W = mg = 50 \times 9.8 = 490 \text{ N}$$

$$N = W + 270 = 490 + 270 = 760 \text{ N}$$

Energy/Work and Momentum/Impulse

Definitions:

The general form of the definition of work done by any force (constant or variable) is as follows:

$$W = \int_{x_i}^{x_f} \vec{F} \cdot d\vec{x}$$

If the force is constant, the above formula can be expressed in the following way:

$$W = \vec{F} \cdot \vec{d} = Fd \cos \theta, \text{ for example, } W = \begin{cases} Fd & \text{if } \theta = 0 \\ 0 & \text{if } \theta = 90^\circ \\ -Fd & \text{if } \theta = 180^\circ \end{cases}$$

Power: the time rate at which a force does work on an object. It measures how fast a force can do work on an object.

$$P = \frac{W}{t}$$

Relationship between work and energy:

Work done by external forces = change in energy of the system

$$W = TE_f - TE_i$$

Note: this relationship can be expressed in different ways depending on how one defines the external forces or the system in study.

Kinetic Energy (KE or E_k):

$$KE = \frac{1}{2} mv^2$$

Potential Energy (PE or E_p): change in potential energy is defined as the negative of the work done by a conservative force. A conservative force is a force whose work does not depend on the path an object follows, such as elastic force, gravity, etc. This can be expressed as:

$$\Delta PE = -W$$

Applying this definition to different situations and forces, we can get the following formulas:

Gravitational Potential Energy: (Gravitational PE):

$$PE_g = mgh$$

Force in a spring:

$\vec{F}_s = -k\vec{x}$, where x is the stretched or compressed distance from the equilibrium position of a spring or other elastic material.

Elastic Potential Energy (Elastic PE):

$$PE_e = \frac{1}{2}kx^2$$

Mechanical Energy (E_M or ME):

$$E_M = PE + KE = PE_g + PE_e + KE$$

The Law of Energy Conservation:

Energy cannot be created or destroyed. It can only be transferred from one object to another, or transformed from one type to another.

For a closed system where there is no non-conservative forces doing work to the system, the total mechanical energy is conserved, which is:

$$TE_{M,i} = TE_{M,f}$$

or:

$$PE_i + KE_i = PE_f + KE_f$$

However, if there are non-conservative force(s) doing work to the system, the total mechanical energy is not conserved any more, but the difference between the total initial mechanical energy and the total final mechanical energy is just the work done by the non-conservative force(s), which is just the “Relationship between Work and Energy”:

$$W = TE_f - TE_i$$

Synthesis Example:

A block (not attached to the spring) with mass $m=0.5\text{kg}$ is placed against a spring on a frictionless incline with angle $\theta = 45^\circ$. The spring constant of the spring is $k = 2000\text{N/m}$. The spring is compressed 16cm and then released. What is the highest distance the block can travel from the release point?

Solution:

If we choose the releasing point as the reference position. Right before the block is released from the spring, the total mechanical energy at that point is as follows:

$$\begin{aligned}
 E_{ik} &= 0 \\
 E_{ip,g} &= 0 \\
 E_{ip,e} &= \frac{1}{2} kx^2 = 0.5 \times 2000 \times 0.16^2 = 25.6 \text{ J}
 \end{aligned}$$

Supposed the block travels x meters to the highest position, then at the highest position:

$$\begin{aligned}
 E_{fk} &= 0 \\
 E_{fp,e} &= 0 \\
 E_{fp,g} &= mgx \sin 45^\circ = 0.5 \cdot 9.8x \cdot 0.866 = 4.24x \text{ J}
 \end{aligned}$$

According to the law of energy conservation:

$$TE_{M,i} = TE_{M,f}$$

$$\begin{aligned}
 \text{i.e. } E_{ik} + E_{ip,g} + E_{ip,e} &= E_{fk} + E_{fp,g} + E_{fp,e} \\
 4.24x &= 25.6
 \end{aligned}$$

$$\text{so } x = 6.03 \text{ m}$$

Momentum and Impulse

$$\text{Momentum: } \vec{P} = m\vec{v}$$

$$\text{Impulse: } \vec{J} = \int_{t_1}^{t_2} \vec{F}_{(t)} dt, \text{ if } \vec{F} \text{ is constant, } \vec{J} = \vec{F}t$$

Newton's 2nd Law (Relationship between momentum and impulse):

$$\vec{J} = \vec{F}t = \Delta\vec{P} = \vec{P}_f - \vec{P}_i = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i)$$

Synthesis Example:

A 0.2kg ball traveling straight with an initial speed of 5m/s bounces off (reverse the original straight path) a wall and comes back in the opposite direction with a speed of 3m/s in 0.1s. (a) What is the impulse acting on the ball? (b) What is the average force exerted by the wall on the ball in this process?

Solution:

(a) The impulse acting on the ball by the wall:

$$\vec{J} = \Delta\vec{P} = m(\vec{v}_f - \vec{v}_i)$$

Choosing the original traveling direction as positive, we get:

$$\vec{J} = 0.2(-3 - 5) = -1.6 \text{ kg} \cdot \text{m/s}$$

so the impulse acting on the ball is 1.6 kg.m/s in the opposite direction.

(b) Since:

$$\begin{aligned}
 J &= Ft \\
 F &= \frac{J}{t} = \frac{1.6}{0.1} = 16 \text{ N}
 \end{aligned}$$

So the average force acting on the ball is 16N, also in the opposite direction.

For a many-body system, if the net external force ($\sum \vec{F}_{\text{ext}}$) is 0, then the total momentum of the system is conserved:

$$\begin{aligned} \text{i.e.} \quad & T\vec{P}_i = T\vec{P}_f \\ \text{i.e.} \quad & \vec{P}_{1i} + \vec{P}_{2i} + \vec{P}_{3i} + \dots = \vec{P}_{1f} + \vec{P}_{2f} + \vec{P}_{3f} + \dots \\ & m_1\vec{v}_{1i} + m_2\vec{v}_{2i} + m_3\vec{v}_{3i} + \dots = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} + m_3\vec{v}_{3f} + \dots \end{aligned}$$

Synthesis Example:

A 1200kg car traveling with a velocity of 15m/s due east collides head-on with a 1400kg car traveling with a velocity of 18m/s due west. The two cars stick and travel together after collision. What is the direction and magnitude of the final velocity of these two cars after collision?

Solution:

External forces such as friction can be ignored in the short time when collision happens, so total momentum is conserved. Choosing east as positive direction, the total initial and final momenta of the 2-car system is given as:

$$TP_i = m_1v_{1i} + m_2v_{2i} = 1200 \times 15 - 1400 \times 18 = -7200 \text{ kg} \cdot \text{m/s}$$

$$TP_f = (m_1 + m_2)v_f = (1200 + 1400)v_f = 2600v_f = -7200 \text{ kg} \cdot \text{m/s}$$

so: $v_f = -2.77 \text{ m/s}$, due west.

Rotational Motion

Torque:

$$\tau = \vec{r} \times \vec{F}$$

Newton's 2nd Law for Rotational Motion:

$$\vec{\tau} = I \vec{\alpha}$$

Here I is the moment of inertia (or, rotational inertia), and it obeys the parallel-axis theorem:

$$I = I_{\text{com}} + Mh^2$$

With a torque acting on a solid object, an object will begin to exhibit rotational motion. Rotational motion with constant angular acceleration is comparable to linear motion with constant linear acceleration:

$$\begin{aligned} \vec{\omega} &= \vec{\omega}_0 + \vec{\alpha}t \\ \vec{\theta} - \vec{\theta}_0 &= \vec{\omega}_0t + \frac{1}{2}\vec{\alpha}t^2 \\ \omega^2 &= \omega_0^2 + 2\vec{\alpha} \cdot (\vec{\theta} - \vec{\theta}_0) \\ \vec{\theta} - \vec{\theta}_0 &= \frac{1}{2}(\vec{\omega}_0 + \vec{\omega})t \end{aligned}$$

$$\vec{\theta} - \vec{\theta}_0 = \vec{\omega}t - \frac{1}{2}\vec{\alpha}t^2$$

Rotational Kinetic Energy:

$$K = \frac{1}{2}I\omega^2$$

PART II FLUIDS

Pressure:

$$p = \frac{F}{A}$$

Pressure in a Fluid:

$$p = p_0 + \rho gh$$

where h is the depth from the surface of the fluid to the point of interest in the fluid.

Archimedes' Principle: The buoyant force exerted on an object by the surrounding fluid is the weight of the volume of fluid displaced:

$$F_b = \rho_{\text{fluid}}gV_{\text{displaced fluid}}$$

Examples:

Study 3 cases of objects in a fluid: floating, suspension, and sinking, assuming only two forces acting on the objects: weight and buoyant force, without external forces acting on the fluid-object system.

Solution:

If an object is floating (partially submerged) in the fluid:

$$F_b = W \text{ (weight of the object)}$$

i.e.

$$\rho_{\text{fluid}}gV_{\text{displaced fluid}} = \rho_{\text{object}}gV_{\text{object}}$$

If the object is fully submerged but suspended within the fluid by a string holding it up so as to prevent it from sinking (note string could also be preventing the object from floating up), we still have:

$$F_b = W \text{ (weight of the object)}$$

i.e.

$$\rho_{\text{fluid}}gV_{\text{object}} = \rho_{\text{object}}gV_{\text{object}}$$

implying:

$$\rho_{\text{fluid}} = \rho_{\text{object}}$$

If the object sinks in the fluid,

$$W > F_b$$

Equation of Continuity:

$$\frac{dV}{dt} = \text{constant}$$

implying:

$$Av = \text{constant}$$

Bernoulli's Equation:

$$p + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$$

Example:

A sphere with a radius 50cm and mass $5 \times 10^3 \text{ kg}$ floats half-submerged in a liquid. What is the density of the liquid?

Solution:

The average density of the sphere can be calculated as follows:

$$\rho_{\text{sphere}} = \frac{m}{V_{\text{sphere}}} = \frac{m}{\frac{4}{3}\pi r^3} = \frac{5 \times 10^3}{\frac{4}{3} \cdot 3.14 \cdot 0.5^3} = 4.78 \times 10^3 \text{ kg/m}^3$$

Since the sphere floats half-submerged,

$$W_{\text{sphere}} = m_{\text{sphere}}g = F_{\text{buoyant}}$$

and

$$V_{\text{submerged}} = \frac{1}{2}V_{\text{sphere}}$$

so:

$$\rho_{\text{sphere}}V_{\text{sphere}}g = \rho_{\text{liquid}}V_{\text{liquid}}g$$

which becomes:

$$\rho_{\text{sphere}}V_{\text{sphere}}g = \frac{1}{2}\rho_{\text{liquid}}V_{\text{sphere}}g$$

therefore:

$$\rho_{\text{liquid}} = 2\rho_{\text{sphere}} = 2 \times 4.78 \times 10^3 = 9.56 \times 10^3 \text{ kg/m}^3$$

PART III WAVES

Sinusoidal Waves:

$$y_{(x,t)} = y_m \sin(kx - \omega t)$$

$$\text{where } k = \frac{2\pi}{\lambda}, \omega = 2\pi f = \frac{2\pi}{T}, v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f.$$

Interference of Waves:

$$Y_{(x,t)} = y_{1(x,t)} + y_{2(x,t)}$$

If $y_{1(x,t)}$ and $y_{2(x,t)}$ are two waves traveling in the same direction on the same string with the same amplitude, frequency and wavelength, but differ in initial phase, then the interference of these two waves will produce a resultant wave as follows:

$$y'_{(x,t)} = 2y_m \cos \frac{\phi}{2} \sin(kx - \omega t + \frac{\phi}{2})$$

If $y_{1(x,t)}$ and $y_{2(x,t)}$ are two waves traveling in opposite directions on the same string with the same amplitude, frequency, wavelength and initial phase. The combination of these two waves will produce a resultant wave as follows:

$$y'_{(x,t)} = 2y_m \sin kx \cos \omega t$$

Example:

A transverse wave traveling along a long string is governed by the equation of $y = 9.0 \sin(0.040\pi x - 2.0\pi t)$, where x and y are expressed in centimeters and t is in seconds. What are (a) the amplitude, (b) the wavelength, (c) the frequency, (d) the speed, (e) the direction of propagation of the wave, and (f) the maximum transverse speed of a particle in the string? (g) What is the transverse displacement at $x = 5 \text{ cm}$ when $t = 0.6 \text{ s}$?

Solution:

(a) The amplitude is $y_m = 9.0 \text{ cm}$.

(b) $\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.04\pi} = 50 \text{ cm}.$

(c) Since $\omega = 2\pi f$, we obtain $f = \frac{\omega}{2\pi} = \frac{2\pi}{2.0\pi} = 1 \text{ Hz}.$

(d) The wave speed is $v = \lambda f = 50 \text{ cm} \times 1 = 50 \text{ cm/s}.$

(e) The wave propagates in the +x direction, since the argument of the trigonometric function is $kx - \omega t$.

(f) The maximum transverse speed (found from the time derivative of y) is

$$u_{\max} = 2\pi f y_m = 2\pi \cdot 1\text{s}^{-1} \cdot 9\text{cm} = 56.52 \frac{\text{cm}}{\text{s}}$$

(g) $y(x=5\text{ cm}, t=0.6\text{ s}) = (9.0\text{ cm}) \sin[0.040\pi(5) - 2.0\pi(0.6)] = 0\text{ cm}$.

The Doppler Effect:

$$f' = f \frac{v \pm v_D}{v \pm v_S}$$

where the sign before v_D is positive if the detector moves towards the source, and the sign before v_S is positive if the source moves away from the detector.

PART IV ELECTRICITY and MAGNETISM

Electric Field

Coulomb's Law:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

q_1 and q_2 are the amounts of charge on two point charges (“+” or “-”), which are separated by a distance of r . The constant $\epsilon_0 = 8.85 \times 10^{-12} \text{C}^2/\text{N} \cdot \text{m}^2$ is the permittivity constant, $k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \text{N} \cdot \text{m}^2/\text{C}^2$, and \hat{r} is a unit vector in the radial direction along an imaginary line joining one charge (q_1) to the other (q_2).

Definition of the Electric Field:

$$\vec{E} = \frac{\vec{F}}{q}$$

In applying Coulomb's Law to the above formula, we obtain the following:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

This provides us with a formula for the electric field generated by a point charge. If there is more than one point charge in a given region space, the total electric field at an arbitrary point can be calculated using the principle of superposition:

$$\vec{E} = \sum_{i=1}^n \vec{E}_i = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2} \hat{r}_1 + \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_2^2} \hat{r}_2 + \frac{1}{4\pi\epsilon_0} \frac{q_3}{r_3^2} \hat{r}_3 + \dots$$

Synthesis Example:

Four equal positive charges are located at the corners of a square ($a=4\text{cm}$). What is the total electric field at the center of the square?

Solution:

The electric field generated by each charge at the corner is:

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

Since the amount of charge and distance from each charge to the center are all the same, the magnitude of the electric field generated by each charge at the corner is also the same. However, the directions of the contributing electric fields are all different. The directions of the electric fields generated by pairs of charges diagonally situated about the center of the square oppose each other exactly, and therefore, they cancel one other. We have two such pairs to consider, and so finally, the total electric field generated by four identical charges at the corners of the square is 0 if computed at the center.

If we consider a continuous charge distribution, the summation is replaced by an integration, such that:

$$\vec{E} = \sum_{i=1}^n \vec{E}_i = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{r}$$

If the electric field is specified, the force acting on a point charge that is subject to the field can be calculated using the following formula:

$$\vec{F} = q\vec{E}$$

Gauss' Law:

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Φ is the net electric flux through the closed surface of area A . The direction of $d\vec{A}$ points outward on the closed Gaussian surface. In other words, $d\vec{A}$ is an outward normal unit vector.

If a convenient Gaussian surface can be constructed so that the electric field everywhere on that surface is the same in magnitude and the direction of $d\vec{A}$ coincides with that of \vec{E} everywhere on the surface, then the dot product $\vec{E} \cdot d\vec{A}$ will yield the same quantity everywhere. We can then bring the magnitude of \vec{E} out in front of the integral sign (since it is uniform over the Gaussian surface) and rather simply integrate over the total surface area. Subsequently, we divide through by this quantity obtained from integration and read off the strength of our electric field. Note: we had to know the direction of \vec{E} to begin with in order to calculate its magnitude. This is typically achieved by invoking symmetry arguments.

Synthesis Example:

A solid non-conducting sphere of radius $R = 50\text{ cm}$ is uniformly charged with a density $\rho = 2\mu\text{C/m}^3$, (a) what is the electric field 20 cm from the center of the sphere? (b) What is the electric field 60 cm from the center?

Solution:

Using Gauss' law:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

(a) For $r = 20\text{ cm}$, we are inside the sphere, and the Gaussian surface is just the spherical surface 20 cm away from the center, and the quantity q is the total amount of charge enclosed by this Gaussian surface.

Therefore,

$$E \cdot 4\pi r^2 = \frac{\rho V}{\epsilon_0}$$

$$E \cdot 4\pi \cdot 0.2^2 = \frac{2 \times 10^{-6} \times \frac{4}{3}\pi 0.2^3}{\epsilon_0}$$

$$E = 15000\text{ N/C}$$

(b) In considering $r = 60\text{ cm}$, we find ourselves outside the spherical distribution of charge, and the Gaussian surface now is the spherical surface 60 cm radially away from the center, and the quantity q happens to be the entire charge of our sphere as it still has to be the total amount of charge enclosed by our Gaussian surface. (**Note that the radius of the Gaussian surface differs from that of the surface extent of our charge distribution**).

Therefore,

$$E \cdot 4\pi r^2 = \frac{\rho V}{\epsilon_0}$$

$$E \cdot 4\pi \cdot 0.6^2 = \frac{2 \times 10^{-6} \times \frac{4}{3} \pi 0.5^3}{\epsilon_0}$$

$$E = 26144 \text{ N/C}$$

Electric Potential Energy: The general definition of electric potential energy is the same as the definition of other types of potential energy:

$$\Delta PE = \Delta U = U_f - U_i = -W$$

where W is the work done by the electrostatic force on a point charge during a displacement move from position i to f .

At infinity, the potential energy is defined to be 0, and so by applying the definition above, the potential energy at any position can be expressed as:

$$U = -W_\infty$$

where W_∞ is the work done by the electrostatic force on the point charge as the charge is displaced from infinity to the particular point of interest. Note that the work done on the same point charge in displacing it from the particular point of interest out to infinity is simply $-W_\infty$.

Electric Potential (V): This quantity is defined as the electric potential energy per unit charge:

$$V = \frac{U}{q}$$

Electric Potential Difference (ΔV):

$$\Delta V = V_f - V_i = \frac{\Delta U}{q}$$

If a point charge is displaced from one point (i) to another (f), the electric potential difference between these two points can also be calculated using the electric field \vec{E} as follows:

$$\Delta V = V_f - V_i = - \int_i^f \vec{E} \cdot d\vec{s}$$

Here $d\vec{s}$ is the path element from i to f . If a charge is brought from infinity (where $V=0$) to some point, then the potential at that point is just:

$$V = - \int_i^f \vec{E} \cdot d\vec{s}$$

Example:

The magnitude E of an electric field depends on the radial distance r according to $E = A/r^3$, where A is a constant with the units of volt·cubic meter. In terms of A , what is the magnitude of the electric potential difference between $r = 1.00$ m and $r = 4.00$ m?

Solution:

$$\Delta V = - \int_1^4 \frac{A}{r^3} dr = \frac{A}{2} \left(\frac{1}{4^2} - \frac{1}{1} \right) = -\frac{15A}{32} = -0.47A$$

Using this method to calculate the potential of a point charge, we find:

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

If there is a scattered distribution of multiple point charges, the total electric potential at a given point can be found by summing up the potential generated by each point charge:

$$V = \sum_{i=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i}$$

For a continuous charge distribution, the summation becomes an integration once again:

$$V = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r}$$

Capacitance: The capacitance C of a capacitor is defined as:

$$q = CV$$

or,

$$C = \frac{q}{V} \text{ and } V = \frac{q}{C}$$

The capacitances of different types of capacitors can be determined through the use of Gauss' Law and the definition of potential difference in terms of the electric field. Here are some examples of such:

The capacitance of a parallel-plate capacitor with an area of A on any one of the plates and having a separation distance d is:

$$C = \frac{\epsilon_0 A}{d}$$

The capacitance of a cylindrical capacitor of length L , inner radius a , and outer radius b is:

$$C = 2\pi\epsilon_0 \frac{L}{\ln\left(\frac{b}{a}\right)}$$

The capacitance of a spherical capacitor with inner radius a and outer radius b :

$$C = 4\pi\epsilon_0 \frac{ab}{b - a}$$

The capacitance of an isolated sphere of radius R is:

$$C = 4\pi\epsilon_0 R$$

Capacitance of capacitors in series:

$$C_{eq} = \frac{1}{\sum_{j=1}^n \frac{1}{C_j}}$$

Capacitance of Capacitors in parallel:

$$C_{eq} = \sum_{j=1}^n C_j$$

Potential Energy (U) stored in a capacitor:

$$U = \frac{q^2}{2C} = \frac{1}{2} CV^2$$

Energy density of the electric field:

$$u = \frac{1}{2} \epsilon_0 E^2$$

Electrical Circuits

Series Circuit:

$$R_{eq} = \sum_{i=1}^n R_i = R_1 + R_2 + R_3 + \dots$$

Parallel Circuits:

$$\frac{1}{R_{eq}} = \sum_{i=1}^n \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots$$

Power:

$$P = IV = I^2 R = \frac{V^2}{R}$$

RC Circuits:

(i) Charging a capacitor:

$$q = C\mathcal{E} \left(1 - e^{-\frac{t}{RC}} \right)$$

(ii) Discharging a capacitor

$$q = q_0 e^{-\frac{t}{RC}}$$

Magnetism

Magnetic Force on a test charge:

$$\vec{F}_B = q \vec{v} \times \vec{B}$$

Magnetic Force on a Current-Carrying Wire:

$$\vec{F}_B = I \vec{L} \times \vec{B}$$

Torque on a Current-Carrying Coil:

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

The Biot-Savart Law:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i d\vec{s} \times \hat{r}}{r^2},$$

where

$$\mu_0 = 4\pi \times 10^{-7} \text{ T} \cdot \frac{\text{m}}{\text{A}} \approx 1.26 \times 10^{-6} \text{ T} \cdot \frac{\text{m}}{\text{A}}$$

\vec{ds} is a very small element in which current is flowing.

For a continuous current distribution, the magnetic field can be calculated by carrying out an integration of the above formula.

Ampere's Law:

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 i_{\text{enclosed}}$$

Here, \vec{ds} is a small element of a closed integration loop. This law provides us a way to calculate the magnetic field \vec{B} if we can exploit some kind of symmetry in the current distribution and find a convenient closed loop to serve as the integration path.

NOTE: The application of Ampere's Law is similar to that of Gauss' Law in the context of electric fields. As such, please refer to earlier notes.

Faraday's Law of Induction: A change in magnetic flux through an area that is bound by a closed conducting loop induces a current along the loop.

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

Here, Φ_B is the magnetic flux and is defined as: $\Phi_B = \int \vec{B} \cdot d\vec{A}$, or simply, $\Phi_B = BA$ if \vec{B} is uniform and perpendicular to the area, and \mathcal{E} is an induced voltage, or what is known as an induced emf (electromotive force).

If there exists a coil of N turns instead of a single loop, the induced emf is, in general:

$$\mathcal{E} = -N \frac{d\Phi_B}{dt}$$

Lenz's Law: An induced current has a direction so that the magnetic field due to this induced current opposes the change in magnetic flux. That's also the reason we have a negative sign appearing in Faraday's Law of Induction.

By applying Faraday's Law and Lenz's Law, one can easily determine the inductance of an inductor.

***Example 1 (changing magnetic field):**

A rectangular wire loop of length $L = 40.0$ cm and width $W = 25.0$ cm lies in a magnetic field \vec{B} . What are the (a) magnitude of \mathcal{E} and (b) direction (clockwise or counterclockwise — or “none” if $\mathcal{E} = 0$) of the emf induced in the loop if $\vec{B} = (6 \times 10^{-2} \text{ T/s})\hat{k}$?

Solution:

(a) For $\vec{B} = (6 \times 10^{-2} \text{ T/s})t\hat{k}$,

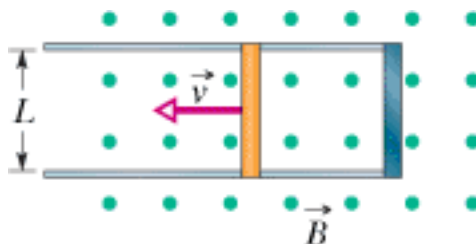
$$\mathcal{E} = \frac{d\Phi_B}{dt} = A \frac{dB}{dt} = (0.400 \text{ m} \times 0.250 \text{ m})(0.0600 \text{ T/s}) = 6.00 \text{ mV},$$

or $|\mathcal{E}| = 6.00 \text{ mV}$.

(b) Using Lenz's law, one can easily see that the direction has to be clockwise.

*Example 2 (changing area):

A metal rod is forced to move with constant velocity \vec{v} along two parallel metal rails, connected with a strip of metal at one end. A uniform magnetic field of magnitude $B = 0.350 \text{ T}$ points out of the page. (a) If the rails are separated by $L = 25.0 \text{ cm}$ and the speed (constant) of the rod is 55.0 cm/s , what emf is generated? (b) If the rod has a resistance of 18.0Ω and the rails and connecting strip have negligible resistance, what is the current in the rod? (c) At what rate is energy being transferred into thermal energy?

Solution:

(a)

$$\mathcal{E} = BLv = (0.350 \text{ T})(0.250 \text{ m})(0.55 \text{ m/s}) = 0.0481 \text{ V}.$$

(b) By Ohm's law, the induced current is

$$i = 0.0481 \text{ V}/18.0 = 0.00267 \text{ A}.$$

By Lenz's law, the current is clockwise.

(c) The power is:

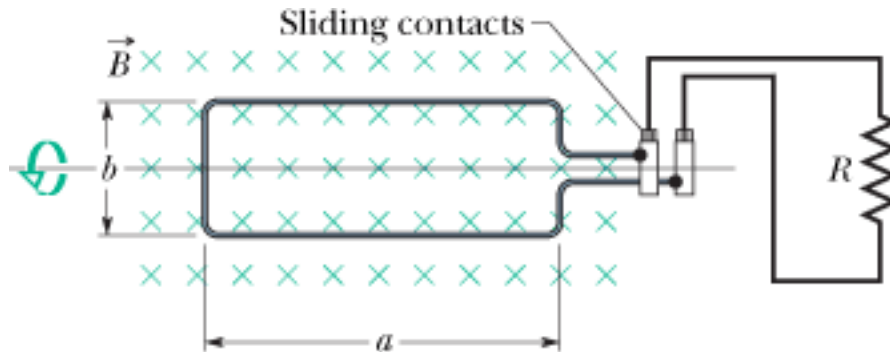
$$P = i^2 R = 0.000129 \text{ W}.$$

*Example 3 (changing angle):

A rectangular coil of N turns having a length a and width b is rotated at frequency f in a uniform magnetic field \vec{B} , as indicated in the figure below. The coil is connected to co-rotating cylinders, against which metal brushes slide to make contact. (a) Show that the

emf induced in the coil is given (as a function of time t) by

$$\mathcal{E} = 2\pi f N a b B \sin(2\pi f t) = \mathcal{E}_0 \sin(2\pi f t).$$



Solution:

It should be emphasized that the result, given in terms of $\sin(2\pi f t)$, could as easily be given in terms of $\cos(2\pi f t)$ or even $\cos(2\pi f t + \phi)$ where ϕ is a phase constant. The angular position θ of the rotating coil is measured from some reference line (or plane), and which line one chooses will affect whether the magnetic flux should be written as $BA \cos \theta$, $BA \sin \theta$ or $BA \cos(\theta + \phi)$. Here, our choice is such that $\Phi_B = BA \cos \theta$. Since the coil is rotating steadily, θ increases linearly with time. Thus, $\theta = \omega t$ (equivalent to $\theta = 2\pi f t$) if θ is understood to be in radians (and ω would be the angular velocity). Since the area of the rectangular coil is $A=ab$, Faraday's law leads to

$$\mathcal{E} = -N \frac{d(BA \cos \theta)}{dt} = -NBA \frac{d \cos(2\pi f t)}{dt} = N B a b 2\pi f \sin(2\pi f t)$$

which is the desired result. The second way this is written ($\mathcal{E}_0 \sin(2\pi f t)$) is meant to emphasize that the voltage output is sinusoidal (in its time dependence) and has an amplitude of $\mathcal{E}_0 = 2\pi f N a b B$.

***The above 3 examples are cited from Chapter 30 of Fundamentals of Physics by Halliday & Resnick (9th Edition)**

LC Charge and Current Oscillations

$$L \frac{d^2 q}{dt^2} + \frac{1}{C} q = 0$$

Alternating Currents; Forced Oscillations:

$$\mathcal{E} = \mathcal{E}_m \sin \omega t$$

and the driven current is:

$$i = I \sin (\omega t - \phi)$$

where $\omega = \frac{1}{\sqrt{LC}}$

Maxwell's Equations

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{A}$$

A. DIMENSIONAL ANALYSIS

Dimensional analysis is a problem solving method in which units are used to guide the calculation. Use the following steps to help you set up your calculation correctly:

1. Identify the given information (magnitude and unit)
2. Identify the desired unit in the final answer (the new unit)
3. Determine the conversion factors needed and write it as a ratio
4. Set up your solution so that the given unit cancels out and the desired unit remains

$$\text{given unit} \times \frac{\text{new unit}}{\text{given unit}}$$

Example:

Convert 450 g to kilogram (kg).

1. Given information: 450 g
2. Desired unit: kg
3. Conversion factor: 1 kg = 1000 g (metric prefix kilo, k = 1000)
4. Setup:

$$? \text{ kg} = 450 \text{ g} \times \frac{1 \text{ kg}}{1000 \text{ g}} = 0.45 \text{ kg}$$

Note that 1000 g is in the denominator of the conversion factor so that the gram unit cancels out the gram unit in the given; only the unit kg remains when the calculation is carried out.

Note that the same type of setup is used when doing calculations involving the mole and stoichiometric calculations in sections B, C and D of this guide.

A. THE MOLE CONCEPT

The mole concept is probably the most important concept in chemistry and usually the one students have the most difficulty with! The mole concept is the bridge or the link between the microscopic world of atoms, molecules and ions and the macroscopic world in which we measure in grams.

$$1 \text{ mole of any substance} = 6.022 \times 10^{23} \text{ units of that substance}$$

Always remember that the mole is simply a number just like the term “dozen” simply means 12. It’s just that it’s a very, very large number. The **molar mass** is

the mass of one mole (6.022×10^{23} units) of any substance in grams. The molar mass is calculated by summing the masses of the atoms in one unit of the substance. The atomic masses are obtained from the Periodic Table.

Example 1:

Calculate the molar mass of water, H_2O .

Solution:

Sum the masses of 2 hydrogen atoms and one oxygen atom. From the periodic table, the atomic mass of H = 1.01 g and O = 16.00 g.

$$\text{Molar mass of } H_2O = 2(1.01 \text{ g}) + 16.00 \text{ g} = 18.02 \text{ g}$$

Example 2:

What is the mass of 1.75 moles of H_2O ?

Solution:

We can calculate the mass of 1 mole of H_2O (called the molar mass: see example 1). We can then use the molar mass to calculate how much 1.75 moles would weigh in grams. This is an example of the dimensional analysis introduced in topic A.

$$g \text{ } H_2O = 1.75 \text{ mol } H_2O \times \frac{18.02 \text{ g}}{1 \text{ mol } H_2O} = 31.5 \text{ g}$$

Similarly, by using the inverse of the conversion factor, we can convert grams of a substance to moles of the substance.

Example 3:

How many water molecules are in 1.75 moles of H_2O ?

Solution:

We can use the definition of the mole to determine how many units (in this example, molecules) are in a given number of moles.

$$\begin{aligned} \text{molecules } H_2O &= 1.75 \text{ mol } H_2O \times \frac{6.022 \times 10^{23} \text{ molecules}}{1 \text{ mol } H_2O} \\ &= 1.05 \times 10^{24} \text{ } H_2O \text{ molecules} \end{aligned}$$

Similarly, by using the inverse of the conversion factor, we can convert the number of units of a substance to moles of a substance.

B. STOICHIOMETRY

This is the calculation of the mass relationships between any two substances in a chemical reaction. The quantity of one substance (the given) is used to calculate the quantity of a second substance (the unknown) using the following steps:

<i>Molar mass of known</i>	<i>Mole ratio from equation</i>	<i>Molar mass of unknown</i>
Mass of known (g) → moles of known → moles of unknown → mass of unknown (g)		

The information above the arrows show the conversion factor required for each step.

Example 4:

Nitrogen (N₂) and hydrogen (H₂) react under certain conditions to form ammonia (NH₃) according to the following equation: N₂ (g) + 3 H₂ (g) → 2 NH₃ (g).

- (a) How many moles of hydrogen are required to completely react with 2.50 moles of nitrogen?
 (b) How many grams of ammonia can be formed from 6.35 g of hydrogen?

Solution:

- (a) We are given moles of the known (2.50 mol nitrogen) and we asked to find moles of the unknown (hydrogen) therefore we only need the second step in the diagram (moles of known to moles of unknown using the mole ratio from the equation). According to the equation, 3 mol of hydrogen react with 1 mol of nitrogen (3 and 1 are the coefficients in the balanced chemical equation).

$$\text{mol H}_2 = 2.50 \text{ mol N}_2 \times \frac{3 \text{ mol H}_2}{1 \text{ mol N}_2} = 7.50 \text{ mol H}_2$$

- (b) We are given grams of the known (6.35 g hydrogen) and asked to find grams of the unknown (ammonia) therefore we need all the steps in the diagram above. First we must convert the grams of hydrogen to moles of hydrogen using the molar mass of hydrogen; next we must use the mole ratios from the equation to calculate the moles of ammonia that would be formed; lastly, we must convert the moles of ammonia to grams of ammonia using the molar mass of ammonia.

$$g \text{ NH}_3 = 6.35 \text{ g H}_2 \times \frac{1 \text{ mol H}_2}{2.02 \text{ g}} \times \frac{2 \text{ mol NH}_3}{3 \text{ mol H}_2} \times \frac{17.04 \text{ g}}{1 \text{ mol NH}_3} = 35.7 \text{ g NH}_3$$

C. LIMITING REACTANT CALCULATIONS

Stoichiometric calculations like the ones in part C usually have only one piece of known information. This assumes that the other reactants in the equation are present in sufficient quantities to allow the known to completely react. But what happens when this is not the case? *From your point of view, you should recognize a limiting reactant problem by the fact that the problem has two known substances!* The limiting reactant is the one that will get used up in the reaction; some of the other reactant (the excess reactant) will be left over (unused) when the reaction stops. How much product can be made will therefore depend on the limiting reactant. These calculations require us to identify which reactant is limiting and then use that substance to calculate the amount of product that can be formed.

There is more than one way to approach this calculation: my preferred method is as follows:

1. Calculate the moles of product that can be formed from reactant A
2. Calculate the moles of the same product that can be formed from reactant B
3. Compare the results from step 1 and 2: whichever reactant forms the ***smaller*** amount of product is the limiting reactant and that amount of product will be the amount that can actually be formed from the reaction.

Example5:

Nitrogen (N₂) and hydrogen (H₂) react under certain conditions to form ammonia (NH₃) according to the following equation: $\text{N}_2 (\text{g}) + 3 \text{H}_2 (\text{g}) \rightarrow 2 \text{NH}_3 (\text{g})$.

- (a) How many moles of ammonia can be formed when 2.50 moles of hydrogen are mixed with 0.955 moles of nitrogen ?

Solution:

You should recognize this as a limiting reactant calculation because there are 2 knowns given: 2.50 mol H₂ and 0.955 mol N₂. Treat each reactant as a separate calculation in determining the moles of NH₃ that can be formed.

1. First we calculate the mol NH₃ from mol H₂:

$$\text{mol } NH_3 = 2.50 \text{ mol } H_2 \times \frac{2 \text{ mol } NH_3}{3 \text{ mol } H_2} = 1.67 \text{ mol } NH_3$$

2. Next we calculate the mol NH_3 from mol N_2 :

$$\text{mol } NH_3 = 0.955 \text{ mol } N_2 \times \frac{2 \text{ mol } NH_3}{1 \text{ mol } N_2} = 1.91 \text{ mol } NH_3$$

3. Compare the results: H_2 gives the smaller amount of NH_3 , therefore H_2 is the limiting reactant and 1.67 mol is the amount of NH_3 that can be formed.

D. REACTIONS IN AQUEOUS SOLUTION

A solution is a mixture in which one or more substances (the solute) is/are dissolved in another substance (the solvent). Water is the most common solvent on the planet; solutions in which water is the solvent are called aqueous solution.

The concentration of a solution tells us how much solute is dissolved in a given amount of solution. The most commonly used concentration unit in chemistry is molarity (M).

$$\text{Molarity (M)} = \frac{\text{moles of solute}}{\text{liters of solution}}$$

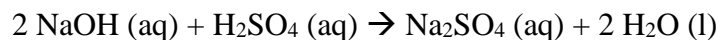
We can calculate the quantities in chemical reactions that occur in solution using the same principles developed in the preceding section. Instead of the amount of the known being given in moles or grams, we are usually given the molarity and volume of the known. This can be used to determine the moles of the known and then the calculation proceeds as before.

Example 6:

What is the molarity of a sodium hydroxide solution (NaOH) if 20.0 mL is neutralized by 32.5 mL of 0.255 M sulfuric acid solution (H_2SO_4)?

Solution:

This is an example of a titration problem. Since it is also a stoichiometric problem, we must start with a balanced chemical equation in order to have the required mole ratio between the known and the unknown.



In this example, the known is sulfuric acid since we know both its molarity (0.255 M) and volume (32.5 mL). We can determine the moles of sulfuric acid by rearranging the equation above for molarity:

$$\text{Moles of solute} = \text{Molarity} \times \text{Liters of solution}$$

Therefore:

$$\text{mol H}_2\text{SO}_4 = 0.255 \text{ M} \times 0.0325 \text{ L} = 8.29 \times 10^{-3} \text{ mol H}_2\text{SO}_4$$

(Note that we converted the volume from mL to L)

We then use the mole ratio from the equation to find the moles of NaOH:

$$\text{mol NaOH} = 8.29 \times 10^{-3} \text{ mol H}_2\text{SO}_4 \times \frac{2 \text{ mol NaOH}}{1 \text{ mol H}_2\text{SO}_4} = 1.66 \times 10^{-2} \text{ mol NaOH}$$

Lastly, to find the molarity of the NaOH solution, we divide the moles of NaOH we calculated in the previous step by the volume of NaOH stated in the problem:

$$\begin{aligned} \text{Molarity of NaOH} &= 1.66 \times 10^{-2} \text{ mol NaOH} / 0.0200 \text{ L NaOH} \\ &= 0.829 \text{ mol/L or } 0.829 \text{ M} \end{aligned}$$

E. THERMOCHEMISTRY

Thermochemical calculations usually are trying to find the heat energy absorbed or released in physical process (for example, changes of state, dissolving) or chemical processes (reactions). The amount of heat absorbed or released can also be used as the starting information to determine the quantities of substances involved in reactions.

Some important points to remember in these calculations:

1. Heat changes have signs attached to the quantities: the positive and negative signs do not mean that the value is less than or greater than zero! Instead, the **signs tell us the direction of heat flow**. A positive sign means the system is gaining energy (think about making a deposit into your bank account). A negative sign means the system is losing energy, just like a withdrawal from your account has a negative sign in your statement.
2. The heat value given in a thermochemical equation applies to the moles of each reactant and product in the balanced equation.

Example:



- a. -92.4 kJ: the negative sign means heat is released when the reaction occurs going from left to right (an exothermic reaction). The reverse reaction would have an enthalpy value of +92.4 kJ, which means heat is absorbed (the reverse reaction is endothermic).
- b. We can write conversion factors using the enthalpy value of -92.4 kJ and the moles in the equation as follows:

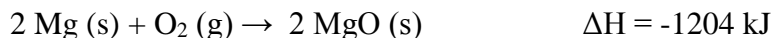
$$\frac{-92.4 \text{ kJ}}{1 \text{ mol N}_2}; \frac{-92.4 \text{ kJ}}{3 \text{ mol H}_2}; \frac{-92.4 \text{ kJ}}{2 \text{ mol NH}_3}$$

Make sure when you write your conversion factors that you include the actual number of moles from the balanced equation. What do these factors tell us? It says that when 1 mole of N₂ reacts according to the equation, 92.4 kJ of heat is released; when 3 moles of H₂ reacts according to this equation, 92.4 kJ of heat is released etc.

C. If you multiply an equation by a factor, you must multiply ΔH by the same factor; if you reverse the equation, you must reverse the sign of ΔH .

Example 7:

Consider the following reaction:



- (a) Is this reaction exothermic or endothermic?
- (b) Calculate the amount of heat transferred when 3.55 g of Mg (s) reacts at constant pressure.
- (c) How many grams of MgO are produced during an enthalpy change of -234 kJ?

Solution:

- a. The reaction is exothermic because the sign of H is negative.
- b. The equation tells us that 1204 kJ of heat is released for every 2 moles of Mg that reacts. Therefore, if we convert 3.55 g Mg to moles (using the molar mass of Mg), we can use the heat: moles ratio from the equation as a conversion factor.

$$? \text{ kJ} = 3.55 \text{ g Mg} \times \frac{1 \text{ mol Mg}}{24.31 \text{ g Mg}} \times \frac{-1204 \text{ kJ}}{2 \text{ mol Mg}} = -87.9 \text{ kJ}$$

C. In this question, we are told how much heat is produced and asked to use this

information to calculate the amount of MgO that is formed. The equation tells us that 1204 kJ of heat is released for every 2 moles of MgO formed: we can use that information as a conversion factor. Think about it this way: if 602 kJ of heat was released (half the amount in the equation), it means 1 mole of MgO (half the amount in the equation) must have formed. We then convert the moles of MgO to grams of MgO using the molar mass of MgO.

Our strategy then:

Convert the given (-234 kJ) → mol MgO using conversion factor of moles: heat from equation → grams MgO using molar mass

$$? \text{ g MgO} = -234 \text{ kJ} \times \frac{2 \text{ mol MgO}}{-1204 \text{ kJ}} \times \frac{40.3 \text{ g MgO}}{1 \text{ mol MgO}} = 15.7 \text{ g MgO}$$

Notice in both parts b and c how the units cancel!

DC CIRCUITS

BASIC CONCEPTS OF ELECTRIC CIRCUIT

Electric Circuit

Definition: An electric circuit is an interconnection of electrical elements. It consists of three basic elements:

- a battery
- a lamp
- a connecting wires.

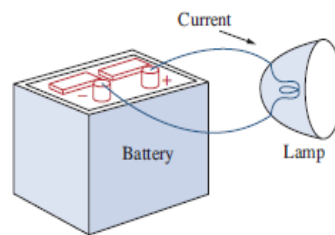


Figure: A Simple Circuit

Charge

Definition: Charge is an electrical property of the atomic particles of which matter consists, measured in coulombs (C). 1 electron (e) has a charge of 1.602×10^{-19} C, or 1 coulomb (C) = 6.24×10^{18} e

Law of conservation

The *law of conservation of charge* states that charge can neither be created nor destroyed only transferred. Thus the algebraic sum of the electric charges in a system does not change.

Electric Current

Definition: Electric current is the time rate of change of charge. Unit of current is Ampere.

Two types of current:

- Direct Current: A direct current (dc) is a current that remains constant with time.

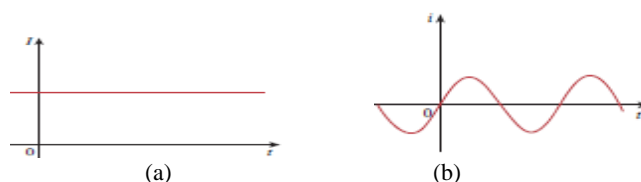


Figure: (a) direct current (b) alternating current

- **Alternative Current:** An alternating current (ac) is a current that varies sinusoidally with time.

Voltage

Definition: Voltage (or potential difference) is the potential energy required to move a unit charge through an element. Unit of voltage is volts (V).

Power and Energy

Definition: Power is the time rate of expending or absorbing energy. Unit of power is watts (W).

$$P = dW/dt$$

where p is power in watts (W), w is energy in joules (J), and t is time in seconds (s).

$$P = vi$$

Example:

How much energy does a 100-W electric bulb consume in two hours?

Solution:

Energy is the capacity to do work, measured in joules (J).

We know, Energy, $w = pt = \text{power} \times \text{time}$ | $p = \text{power} = 100\text{w}$, time $t = 2 \text{ hours} = 7200\text{s}$

$$\begin{aligned} \text{So } w &= 100 \text{ (W)} \times 2 \text{ (h)} \times 60 \text{ (min/h)} \times 60 \text{ (s/min)} \\ &= 720,000\text{J} = 720\text{kJ} \end{aligned}$$

BASIC LAWS

Ohm's Law: A resistor is a passive element in which the voltage v across it is directly proportional to the current i through it. That is, a resistor is a device that obeys Ohm's law,

$$v = iR$$

where R is the resistance of the resistor.

Resistance: A cylinder of length l and cross sectional area A has a resistance:

$$R = \rho \frac{l}{A}$$

where

R = resistance of an element in ohms Ω
 ρ = resistivity of the material in ohm-meters
 l = length of the cylindrical material in meters
 A = cross sectional area of material in meter²

Conductance: Conductance is the ability of an element to conduct electric current. Units of conductance is mhos (σ) or siemens (S).

$$G = \frac{1}{R}$$

Example of Ohm's Law:

An electric iron draws 2 A at 120 V. Find its resistance and conductance.

Solution:

From Ohm's law,

$$v = iR$$

Resistance is

$$R = \frac{v}{i} = \frac{120}{2} = 60\Omega$$

Conductance is

$$G = \frac{1}{R} = \frac{1}{60} = 0.167 \text{ S}$$

Short circuit and Open Circuit:

Definition: A short circuit is a resistor (a perfectly, conducting wire) with zero resistance ($R=0$). Just drawn as a wire (line).

Definition: An open circuit is a resistor with infinite resistance. ($R=\infty$). Just omitted

Branches, Node and Loop:

Branches: A Single two-terminal element in a circuit such as voltage source, resistor, and current source.

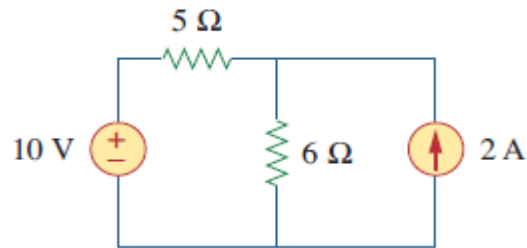
Node: The point of connection between two or more branches. A node is usually indicated by a dot in a circuit.

Loop: Any closed path in a circuit.

Branch Theorem: A network with **b** branches, **n** nodes, and **l** independent loops will satisfy the fundamental theorem of network topology:

$b = l + n - 1$

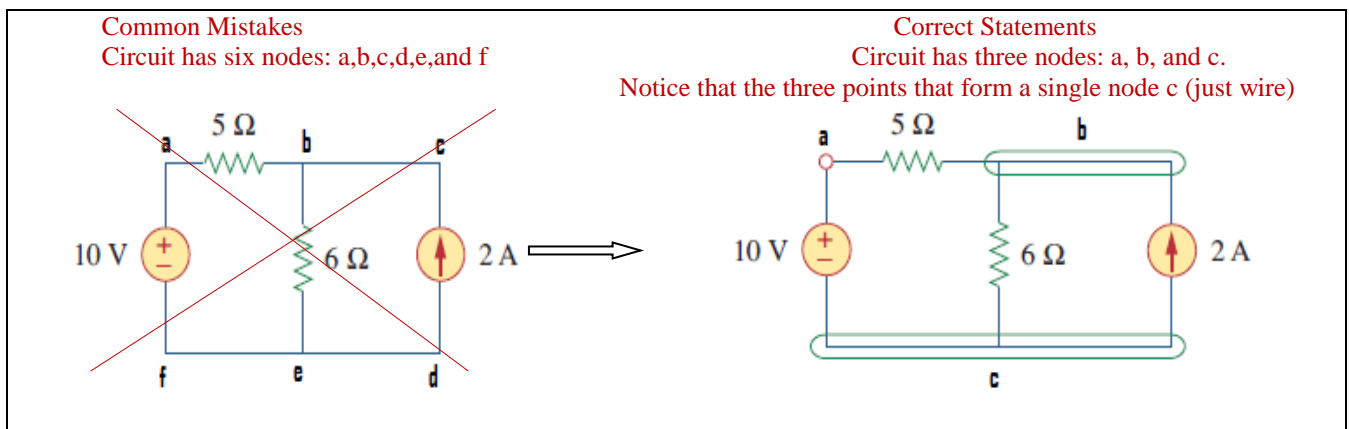
Example: Determine the number of branches and nodes in the following circuit.



Solution:

Branches: The circuit has five branches: 10-V voltage source, the 2-A current source and three resistors.

Nodes: The circuit has three nodes: a, b, and c.



Alternative Method to determine branch:

Using the branch theorem, number of branches, $b = l + n - 1 = 3 + 3 - 1 = 5$

Kirchhoff's Current Law (KCL):

Definition: KCL states that the algebraic sum of currents entering a node (or a closed boundary) is zero.

Example: Consider the node in the following Figure,

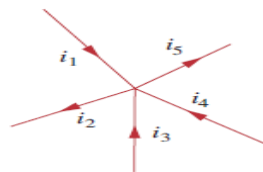


Figure: Currents at a node illustrating KCL

Applying KCL gives,

$i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0$ (Assume incoming current in a node is positive and outgoing current is negative)

Alternative Method:

The sum of the currents entering a node equals the sum of currents leaving the node.

$$i_1 + i_3 + i_4 = i_2 + i_5$$

Kirchhoff's Voltage Law (KVL):

Definition: Kirchhoff's voltage law (KVL) states that the algebraic sum of all voltages around a closed path (or loop) is zero.

Procedure:

- Step 1: Start with any branch in the circuit
- Step 2: Go around the loop either clockwise or counterclockwise.
- Step 3: Sign on each voltage is the polarity of the terminal encountered first as we travel around the loop.
- Step 4: Voltage rises is negative (go from $-$ to $+$ terminal) and drop is positive (go from $+$ to $-$).
- Step 5: Sum of voltage drop is equal to sum of voltage rises.

Example:

Apply KVL in the following circuit:

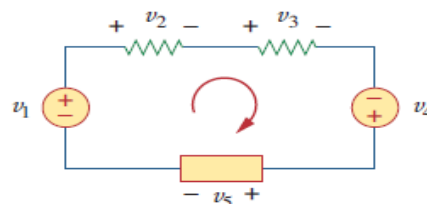


Figure: A single-loop circuit illustrating KVL

- Step 1: Start with the voltage source as first branch, v_1
- Step 2: Go around the loop clockwise
- Step 3: Sign on each voltage is the polarity of the each branch such as v_2 , v_3 , v_5 (Keep in mind current always entering resistance into positive terminal)
- Step 4: $-v_1 + v_2 + v_3 - v_4 + v_5 = 0$
- $v_2 + v_3 + v_5 = v_1 + v_4$ (Sum of voltage drops = Sum of voltage rises)

Common Mistakes:

~~$v_1 + v_2 + v_3 - v_4 + v_5 = 0$ DO NOT AVOID (-) SIGN, when you start from the voltage source v_1~~

or

~~$-v_3 - v_4 + v_5 - v_1 + v_2 = 0$ DO NOT AVOID (+) SIGN, when you start from branch 3.~~

Series Resistors and Voltage divisions:

Definition: The equivalent resistance of any number of resistors connected in series is the sum of the individual resistances.

Example:

The two resistors are in series in the following circuit, because

1. they are connected sequentially end to end
2. the same current i flows in both of them. (Note that when elements are series, the same current flow through them).

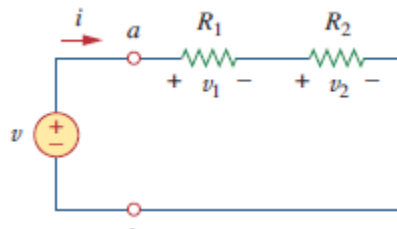


Figure: A single-loop circuit with two resistors in series.

Therefore, the two resistors $R_1 (=1/G_1)$ and $R_2 (=1/G_2)$ can be added by an equivalent resistor R_{eq} and conductance G_{eq} ; that is,

$$R_{eq} = R_1 + R_2 \quad \text{and} \quad G_{eq} = \frac{G_1 G_2}{G_1 + G_2}$$

Voltage divisions:

Definition: The principle of voltage division is that the source voltage v (*above circuit*) is divided among the resistors in direct proportion to their resistances. The circuit in the above figure is also called a voltage divider.

Example: According to voltage divider, voltage across each resistor in the above figure

$$v_1 = \frac{R_1}{R_1 + R_2} v, \quad v_2 = \frac{R_2}{R_1 + R_2} v$$

Parallel Resistors and Current Division:

Definition: The equivalent resistance of two parallel resistors is equal to the product of their resistances divided by their sum.

Example:

When two resistors R_1 ($=1/G_1$) and R_2 ($1/G_2$) are in parallel, their equivalent resistance R_{eq} and equivalent conductance G_{eq} are

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$G_{eq} = G_1 + G_2$$

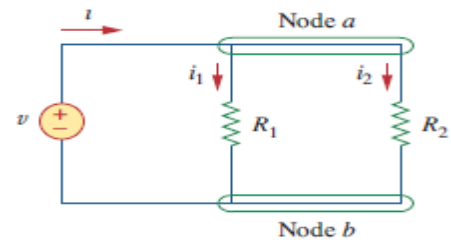


Figure: Two resistors in parallel.

Current Divider:

Definition: The total current i is shared by the resistors in inverse proportion to their resistances. This is known as the principle of current division, and the circuit in the above Figure is known as a current divider.

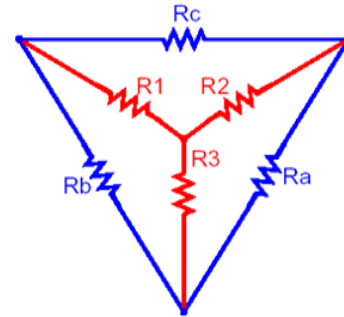
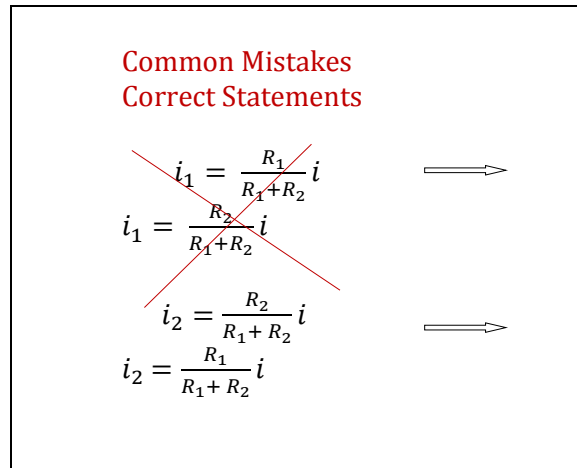
Example:

The current division principle for two resistors in parallel is

$$i_1 = \frac{R_2}{R_1 + R_2} i, \quad i_2 = \frac{R_1}{R_1 + R_2} i$$

Common Mistakes:

$$v_1 = \frac{v}{R_1} \quad v_2 = \frac{v}{R_2}$$



Note: If you want to find the current through the resistor R1, you must multiply by the opposite resistor and total current and divide by the sum of two resistors.

Delta \leftrightarrow Wye Transformations:

Formulas for a wye-to-delta transformation:

$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$$

(Sum of all possible products of **Y** resistors taken two at a time, divided by the opposite **Y** resistor)

Figure: Superposition of Y and Δ networks

$$R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} \text{ (Sum of all possible products of **Y** resistors taken two at a time, divided by the opposite **Y** resistor)}$$

$$R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} \text{ (Sum of all possible products of **Y** resistors taken two at a time, divided by the opposite **Y** resistor)}$$

Formulas for a delta (Δ) –to wye (Y) transformation:

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \text{ (Product of the resistors in the two adjacent Δ branches and divided by the sum of the three Δ resistors)}$$

$R_2 = \frac{R_a R_c}{R_a + R_b + R_c}$ (Product of the resistors in the two adjacent Δ branches and divided by the sum of the three Δ resistors)

$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$ (Product of the resistors in the two adjacent Δ branches and divided by the sum of the three Δ resistors)

Note: The numerator is the same for Y- Δ conversation and the denominator is same for Δ -Y conversation.

Example:

Convert the Δ network in the following figure to an equivalent Y network.

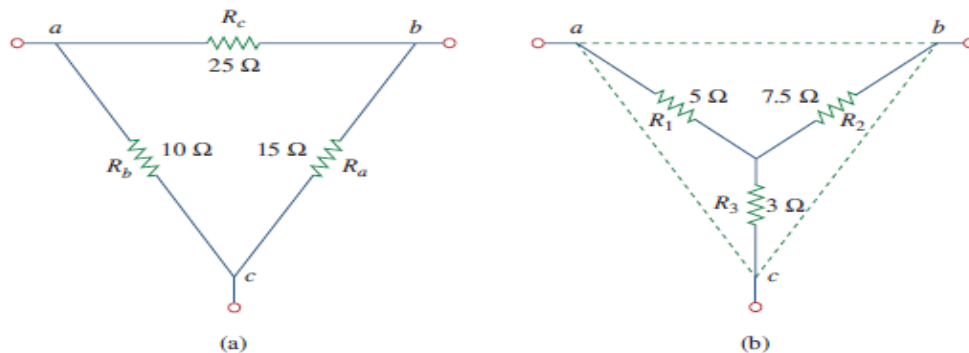


Figure: (a) original Δ network, (b) Y equivalent network.

Solution :

Using Delta to Wye formula,

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} = \frac{(10)(25)}{10 + 25 + 15} = \frac{250}{50} = 5\Omega$$

$$R_2 = \frac{R_a R_c}{R_a + R_b + R_c} = \frac{(15)(25)}{10 + 25 + 15} = \frac{375}{50} = 7.5\Omega$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c} = \frac{(15)(10)}{10 + 25 + 15} = \frac{150}{50} = 3\Omega$$

METHODS OF CIRCUIT ANALYSIS:

- Nodal analysis: A general procedure to find all the node voltages in a circuit. The analysis is based on KCL and Ohm's Law.
- Mesh analysis: A general approach to find mesh currents that circulate around closed paths in the circuit. The analysis is based on KVL and Ohm's Law.

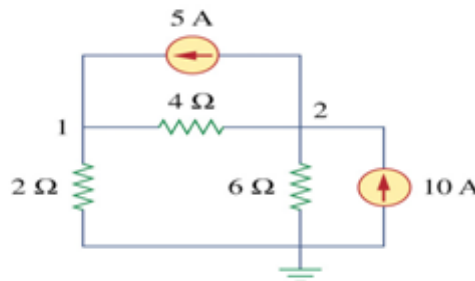
Nodal Analysis:

Steps to determine node voltages:

1. Select a node as the reference node. Assign voltages v_1, v_2, \dots, v_{n-1} to the remaining $n - 1$ nodes. The voltages are referenced with respect to the reference node. (Reference node is commonly called the *ground* since it is assumed to have zero potential).
2. Apply KCL to each of the $n - 1$ non-reference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

Example:

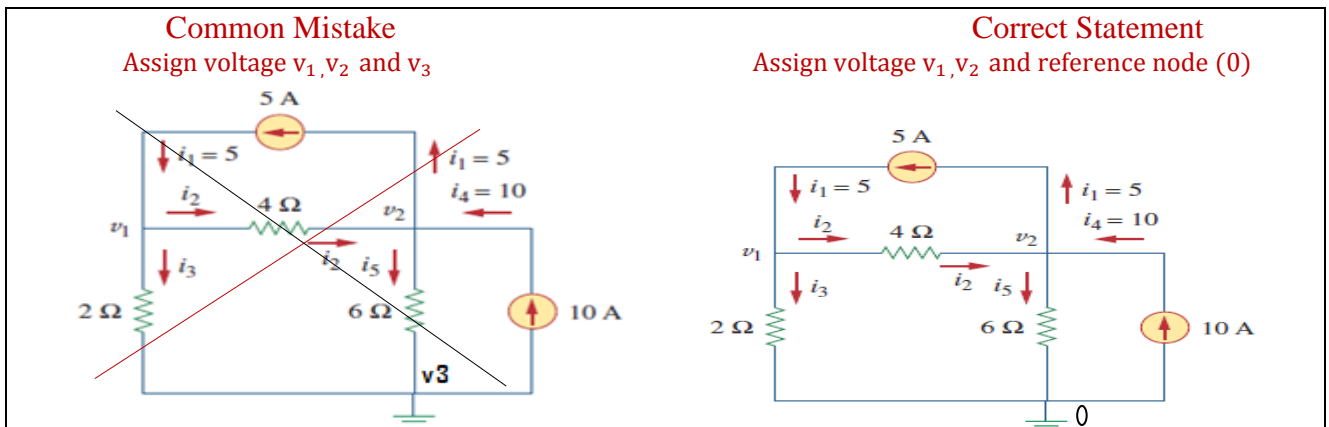
Calculate the node voltages in the following circuit.



Solution:

Step 1: The reference node ($v=0$) is selected, nodes 1 and 2 are assigned voltages v_1 and v_2 respectively.

- Note: The node voltages are defined with respect to the reference node.



Step 2: At node 1, apply KCL and Ohm's law

$$i_1 = i_2 + i_3$$

Common Mistakes:

- $i_1 + i_2 + i_3 = 0$
- (Note that sum of all incoming current is equal to the outgoing current at node 1)
 $i_1 + i_2$

$$5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2} \quad (\text{Note: Current flows from a higher potential to a lower potential in a resistor.})$$

Multiplying each term by 4, we obtain

$$20 = v_1 - v_2 + 2v_1$$

or

$$3v_1 - v_2 = 20 \quad \dots\dots\dots(1)$$

At node 2, we do the same thing and get

$$i_2 + i_4 = i_1 + i_5$$

$$\Rightarrow 5 = \frac{v_2 - v_1}{4} + \frac{v_2 - 0}{6}$$

Multiplying each term by 12 results in

$$3v_1 - 3v_2 + 120 = 60 + 2v_2$$

Or

$$-3v_1 + 5v_2 = 60 \quad \dots\dots\dots(2)$$

Step 3: Solve the unknown node voltages v_1 and v_2 . Using the elimination technique, we add Eqs. (1) and (2)

$$4v_2 = 80 \implies v_2 = 20\text{V}$$

Substituting $v_2 = 20$ in Eq. (1) gives

$$v_1 = 13.33\text{V}$$

Nodal Analysis with Voltage Sources:

- Case 1: The voltage source is connected between a non-reference node and the reference node: The non-reference node voltage is equal to the magnitude of voltage source, and the number of unknown non-reference nodes is reduced by one.
- Case 2: The voltage source is connected between two non-referenced nodes: a generalized node (supernode) is formed.

Example:

Consider the following circuit with a super node.

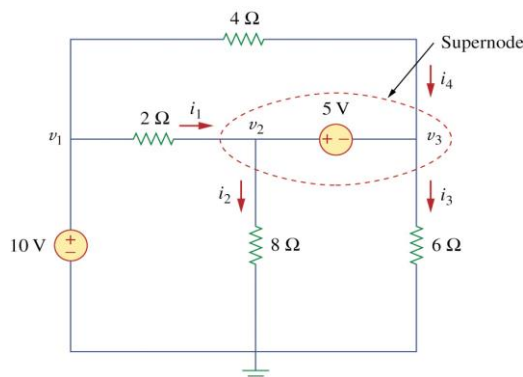


Figure: A circuit with super node

Case 1: 10V source is connected between a non-reference node v_1 and reference node.

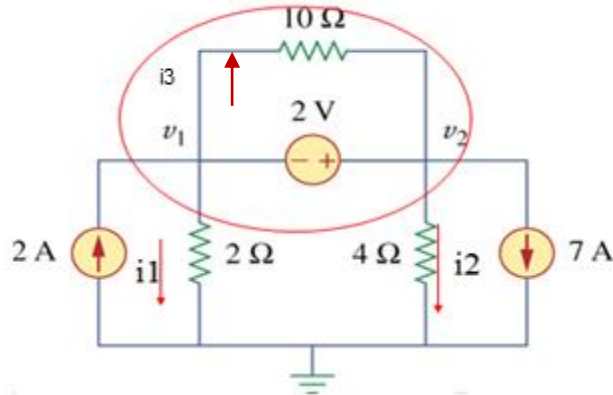
$$\text{So } v_1 = 10\text{ v}$$

Case 2: 5V source is connected between two non-reference nodes v_2 and v_3 . So a super node is formed between v_2 and v_3

- A supernode is formed by enclosing a (dependent or independent) voltage source connected between two non-reference nodes and any elements connected in parallel with it.
- A supernode requires the application of both KCL and KVL.

Example:

Find the node voltages in the following circuit.

Solution:

- Step 1: The super node contains the 2-V source, nodes 1 and 2, and the 10 Ω Resistor
- Step 2: Applying KCL and KVL gives to the super node as shown in the above Figure.

Common Mistake

$$2 = i_1 + i_2 + i_3 + 7$$

Correct Statement

$$2 = i_1 + i_2 + 7$$

Note: Make sure there is no current flow in the super node

$$2 = i_1 + i_2 + 7$$

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7$$

$$v_2 = -20 - 2v_1 \dots \dots \dots (1)$$

Now we apply KVL to the super node

$$v_2 - v_1 = 2 \dots \dots \dots (2)$$

Using the elimination technique, we add Eqs. (1) and (2)

$$v_1 = -7.33\text{V and } v_2 = -5.33\text{V}$$

Common Mistake:

- $v_2 - v_1 + 10i_3 = 0$

(Note that the 10Ω resistor does not make any difference because it is connected across the parallel. So ignore the 10Ω resistor.)

Mesh Analysis: A general approach to find mesh currents that circulate around closed paths in the circuit. It is based on KVL and Ohm's Law.

Steps to Determine Mesh Currents:

1. Assign mesh currents i_1, i_2, \dots, i_n to the n meshes.
2. Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the resulting n simultaneous equations to get the mesh currents.

Example:

Determine the mesh current in the following circuit.

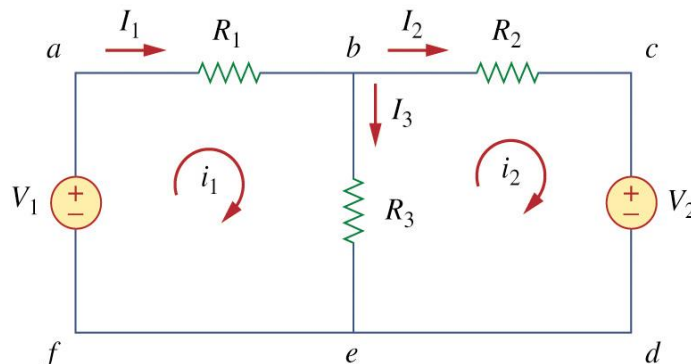


Figure: A circuit with two meshes.

Step 1. Assign mesh currents i_1 and i_2 to meshes 1 and 2.

Step 2: we apply KVL to each mesh.

For mesh 1,

$$-V_1 + i_1 R_1 + (i_1 - i_2) R_3 = 0 \quad \text{or} \quad i_1 R_1 + (i_1 - i_2) R_3 = V_1$$

Alternative:

$$i_1 R_1 + I_3 R_3 = V_1; I_3 = i_1 - i_2$$

For mesh 2,

$$i_2 R_2 + V_2 + (i_2 - i_1) R_3 = 0 \text{ or}$$

Alternative:

$$i_2 R_2 + V_2 - (i_1 - i_2) R_3 = 0 \text{ or}$$

$$i_2 R_2 + V_2 - I_3 R_3 = 0$$

$$i_2 R_2 + (i_2 - i_1) R_3 = V_2$$

Common Mistakes	Correct Statements
$-V_1 + i_1 R_1 + i_1 R_3 = 0$	$-V_1 + i_1 R_1 + (i_1 - i_2) R_3 = 0$
$i_2 R_2 + V_2 + i_2 R_3 = 0$	$i_2 R_2 + V_2 + (i_2 - i_1) R_3 = 0$
$I_3 = i_2 - i_1$	$I_3 = i_1 - i_2$

Step 3: Solve for the mesh currents

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

Mesh Analysis with Current Sources:

Example:

A circuit with a current source in the following Figure.

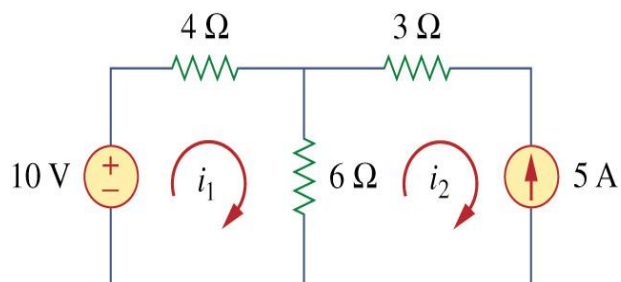


Figure: A circuit with a current source

Procedure:

Case 1:

- Current source exist only in one mesh in the above Figure
So, $i_2 = -5A$
- One mesh variable is reduced

Case 2:

- Current source exists between two meshes, a super-mesh is obtained.

Supermesh:

- A super mesh results when two meshes have a (dependent, independent) current source in common. For example, the following Figure 6A source is common in both meshes.

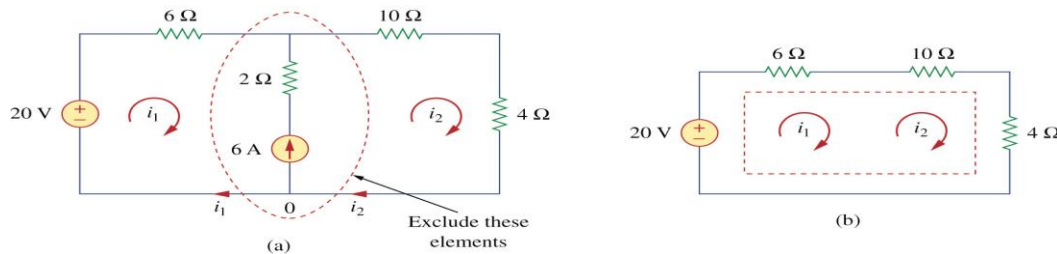


Figure: (a) Two meshes having a current source in common (b) a supermesh, created by excluding the current source.

Procedure:

Step 1: Using KVL and Ohm's law to the super mesh.

Step 2: Using KCL due to the current source.

Example:

Find i_1 and i_2 using mesh analysis in the above Figure

Step1: Applying KVL to the super mesh in above Figure

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

$$6i_1 + 14i_2 = 20 \dots \dots \dots (1)$$

Step 2: Applying KCL to the current source

$$i_2 - i_1 = 6 \dots \dots \dots (2)$$

Common Mistakes	Correct Statements
Step 1: $-20 + 6i_1 + 2(i_1 - i_2) + 10i_2 + 4i_2 + 6 = 0$ Step 2: $i_1 - i_2 = 6$	$-20 + 6i_1 + 10i_2 + 4i_2 = 0$ (Must exclude the current source (6A) and 2Ω resistance) $i_2 - i_1 = 6$ (i_2 is positive because it is same direction of current source and i_1 is negative because it is opposite direction of current source)

Solving Eqs. (1) and (2), we get

$$i_1 = 3.2 \text{ A}, \quad i_2 = 2.8 \text{ A}$$

CIRCUIT THEOREMS:

Superposition Theorem:

Definition: The superposition principle states that the voltage across (or the current through) an element in a linear circuit is the algebraic sum of the voltage across (or the currents through) that element due to EACH independent source acting alone.

Steps to apply superposition principle:

Step 1: Turn off all independent sources except one source. Find the output (voltage or current) due to the active source using KVL, KCL, nodal, or mesh analysis.

Step 2: Repeat step 1 for each of the other independent sources.

Step 3: Find the total contribution by adding algebraically all the contributions due to the independent sources.

Two things have to be keep in mind:

- When we say turn off all other independent sources:
 - Independent voltage sources are replaced by 0 V (short circuit)
 - Independent current sources are replaced by 0 A (open circuit).
- Dependent sources are left intact because they are controlled by circuit variables.

Example:

Use the superposition theorem to find v in the following circuit .

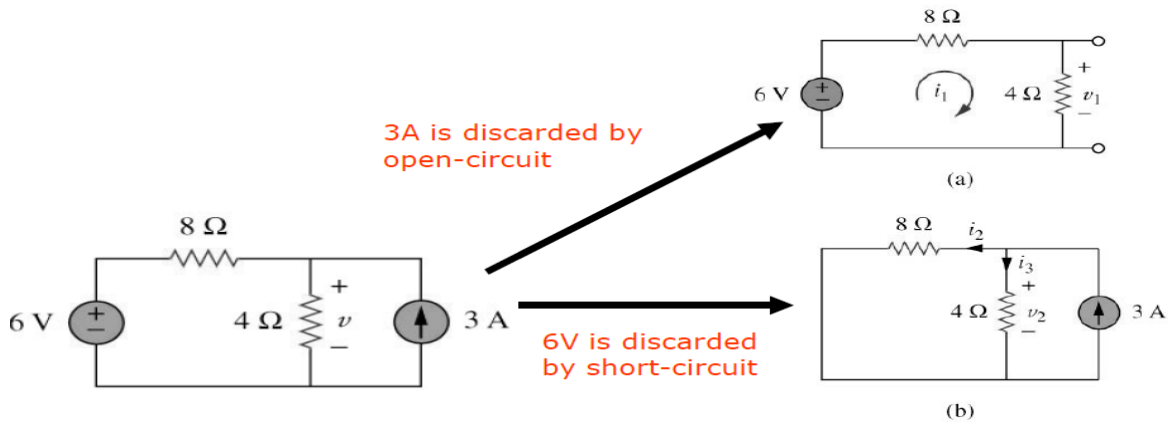


Figure: Example of superposition theorem

Solution:

Let, v_1 and v_2 be the contributions of two sources 6V voltage source and 3A current source.

According to linearity we can write,

$$v = v_1 + v_2$$

Step 1: To obtain v_1 we set current source open circuit or 0A as Figure (a) circuit.

We can use voltage division to get

$$v_1 = \frac{4}{4+8} (6) = 2V \text{ (Note: We can use any method such as voltage division, mesh or nodal analysis to solve for } v_1)$$

Step 2: Repeat step 1 for other independent source (3A) .

To obtain v_2 , we set voltage source short circuit as Figure (b) or 0V

We can use current division to get,

$$i_3 = \frac{8}{4+8} (3) = 2A \text{ (Note: We can use any method such as current division, mesh or nodal analysis to solve for } v_2)$$

Hence,

$$v_2 = 4i_3 = 8V$$

Alternative Method to solve for v_2 : Nodal Analysis

$$\frac{v_2 - 0}{4} + \frac{v_2 - 0}{8} = 3 \text{ (reference node is zero voltage)}$$

Multiply by 8 to both side of the equation

$$\begin{aligned} 2v_2 + v_2 &= 24 \\ v_2 &= 8V \end{aligned}$$

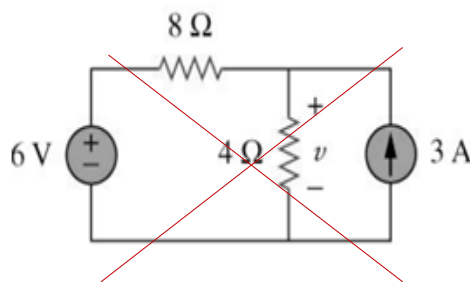
Step 3: Add all the contributions of independent sources algebraically and find total contribution

Finally,

$$v = v_1 + v_2 = 2 + 8 = 10V$$

Common Mistakes

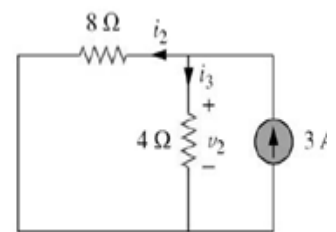
- Consider two sources at a time



- Voltage source must be turn off
- Apply KVL using mesh : $-6 + 8i_1 + 4(3) = 0$

Correct Statements

Consider only one source at a time



Voltage source turn off (short circuit)
Use current division to find $v_2 = \frac{3(8)}{8+4} = 2v$

Source Transformation:

Definition: A source transformation is the process of replacing a voltage source v_s in series with a resistor R by a current source i_s in parallel with a resistor R , or vice versa.

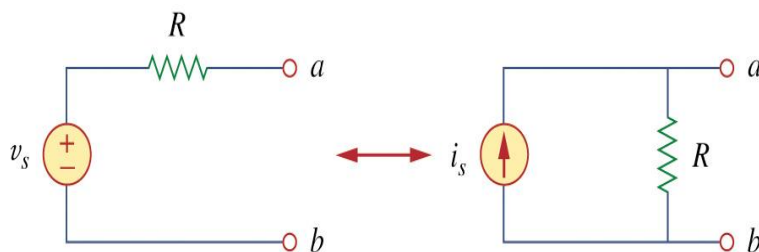
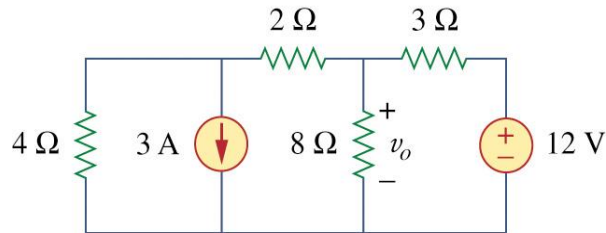


Figure: Transformation of independent sources.

$$v_s = i_s R \text{ or } i_s = \frac{v_s}{R}$$

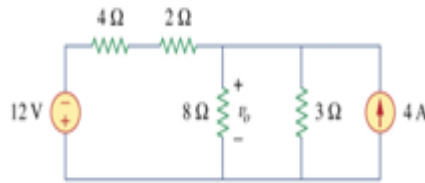
Example:

Find v_o in the circuit shown below using source transformation.



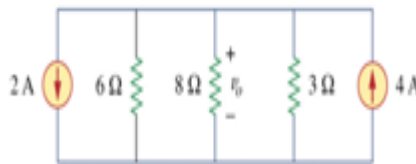
Procedure:

Step 1: First transform the current and voltage sources to obtain the circuit in the following Figure.



Remark: The arrow (\uparrow) of the current source is directed toward the (+) positive terminal of voltage source

Step 2: Combining the 4-Ω and 2 Ω the resistors in series and transforming the 12-V voltage source.



Step 3: Combine the 3-Ω and 6 Ω and resistors in parallel to get 2 Ω. We also combine the 2-A and 4-A current sources to get a 2-A source in the following circuit.



We use current division in the above Figure to get

$$i = \frac{2}{2 + 8} (2) = 0.4 \text{ A}$$

and

$$v_o = 8(0.4) = 3.2 \text{ V}$$

Thevenin's Theorem:

Definition: Thevenin's theorem states that a linear two-terminal circuit (Figure a) can be replaced by an equivalent circuit (Figure b) consisting of a voltage source V_{Th} in series with a resistor R_{Th} ,

where

- V_{Th} is the open circuit voltage at the terminals
- R_{Th} is the input or equivalent resistance at the terminals when the independent source is turned off.

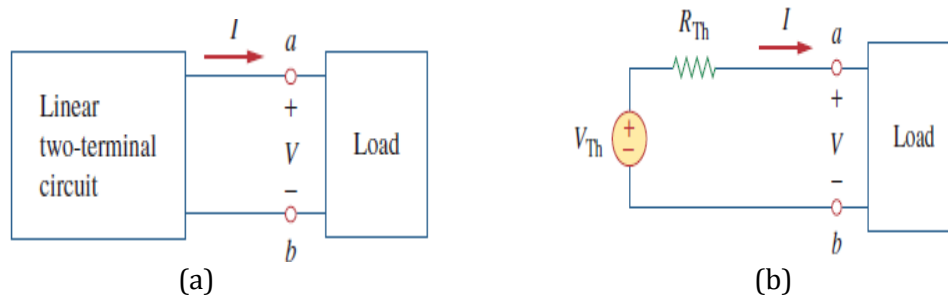


Figure: Replacing a linear two-terminal circuit by its Thevenin equivalent: (a) original circuit, (b) the Thevenin equivalent circuit.

To Find the Thevenin Equivalent circuit:Procedure:

Step 1: Find $R_{th} = R_{in}$ (Input resistance at the terminal $a-b$)

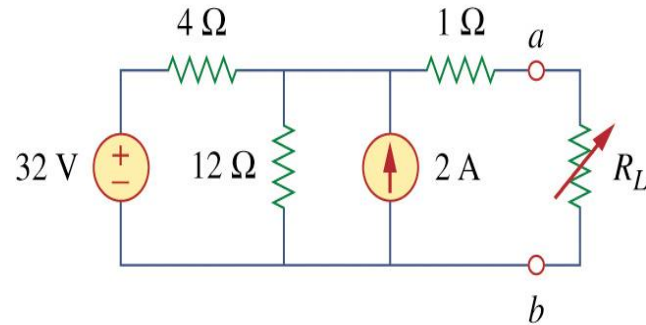
- $a-b$ open circuited (by removing the load)
- Turn of all independent sources (voltage source \rightarrow short and current source \rightarrow open)

Step 2: Find $V_{th} = V_{oc}$: Open circuit voltage at $a-b$ using any method such as Mesh, Nodal analysis, etc.

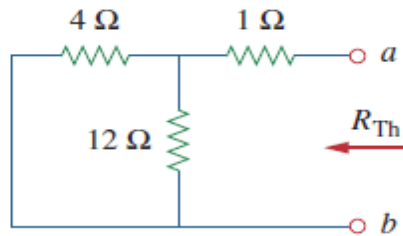
Step 3: To build the Thevenin equivalent circuit using V_{th} and R_{th} .

Example:

Find the Thevenin's equivalent circuit of the following circuit, to the left of the terminals $a-b$. Then find the current through $R_L = 6, 16$, and 36Ω .

**Solution:**

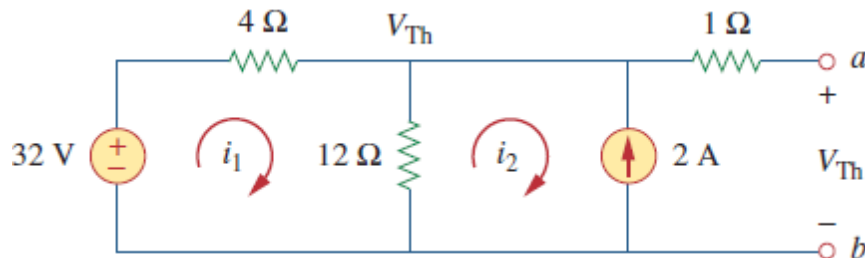
Step 1: Find R_{th} , a - b open circuited (by removing the load).



Note: 32 V voltage source \rightarrow short and 2 A current source \rightarrow open

$$R_{th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\Omega$$

Step 2: Find V_{th} , consider the following circuit (Open circuit voltage at a - b).



Method 1: Mesh analysis

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \quad i_2 = -2A$$

$$\text{So } i_1 = 0.5A$$

$$V_{th} = 12(i_1 - i_2) = 12(0.5 + 2) = 30V$$

Alternative Method 2: Nodal analysis

$$\frac{32 - V_{th}}{4} = \frac{V_{th}}{12}$$

$$V_{th} = 30V$$

Common Mistakes

~~$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \quad i_2 = +2A$$~~

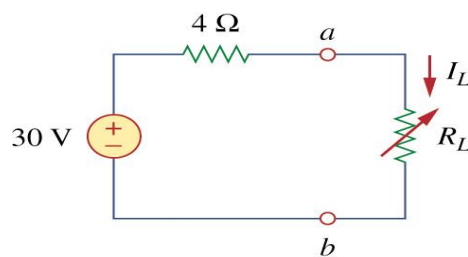
~~Consider current flow through the 1Ω resistance
i.e. $2(1) + V_{th} = 0$~~

Correct Statements

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \quad i_2 = -2A$$

Must be ignore the 1Ω resistance because of a-b is open
No current flow through the 1Ω resistance
 $V_{th} = 12(i_1 - i_2)$

Step 3: The Thevenin equivalent circuit is shown in the following Figure. The current through R_L is



$$I_L = \frac{V_{th}}{R_{th} + R_L}$$

When $R_L = 6$,

$$I_L = 30/10 = 3 \text{ A}$$

When $R_L = 16$,

$$I_L = 30/20 = 1.5 \text{ A}$$

When $R_L = 36$,

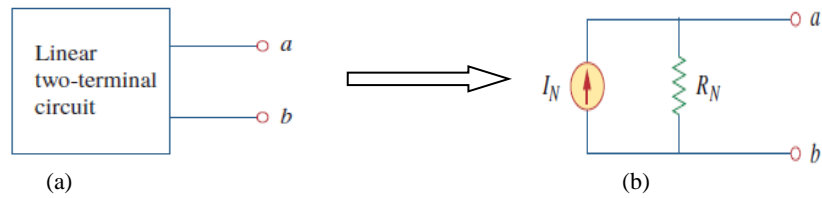
$$I_L = 30/40 = 0.75 \text{ A}$$

Norton's Theorem:

Definition: Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a **current source I_N in parallel with a resistor R_N** .

where

- I_N is the short-circuit current through the terminals.
- R_N is the input or equivalent resistance at the terminals when the independent source is turned off.



To find the Norton Equivalent circuit:

Procedure:

Step 1: Find R_N the same way we find R_{th} using Step 1 of Thevenin Equivalent circuit: $R_N = R_{Th}$

Step 2: Find the short circuit current from a to b :

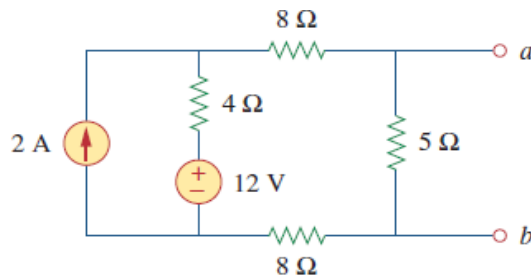
$$I_N = i_{sc} = \frac{V_{Th}}{R_{Th}}$$

Thevenin or Norton equivalent circuit:

- The open circuit voltage $v_{oc} = V_{th}$ across terminals a and b .
- The short circuit current $i_{sc} = I_N$ at terminals a and b
- The equivalent or input resistance $R_{in} = R_{th} = \frac{V_{th}}{R_{th}} = R_N$ at terminals a and b when all independent source are turn off.

Example:

Find the Norton equivalent circuit of the circuit in the following Figure.

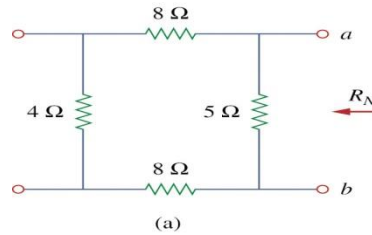


Solution:

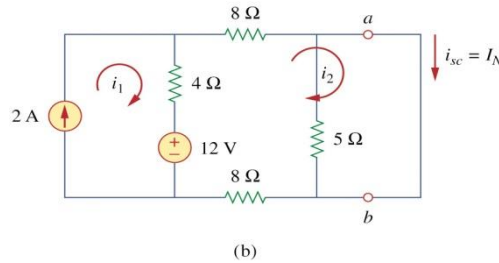
Step 1: To find R_N in the following Figure

$$R_N = 5 \parallel (8 + 4 + 8)$$

$$= 5 \parallel 20 = \frac{20 \times 5}{25} = 4 \Omega$$



Step 2: To find I_N in the following Figure. (short circuit the terminals a and b)



Applying mesh analysis, we obtain

Mesh 1:

$$i_1 = 2A \dots \dots \dots (1)$$

Mesh 2:

$$-12 + 4(i_2 - i_1) + 8i_2 + 8i_2 = 0 \dots \dots \dots (2)$$

$$20i_2 - 4i_1 = 12$$

Common Mistakes	Correct Statements
$i_1 = -2A$	$i_1 = 2A$
$-12 + 4(i_2 - i_1) + 8i_2 + 5i_2 + 8i_2 = 0$	$-12 + 4(i_2 - i_1) + 8i_2 + 8i_2$
$= 0$	
Consider 5 Ω resistance is wrong	Must be ignore 5 Ω resistance

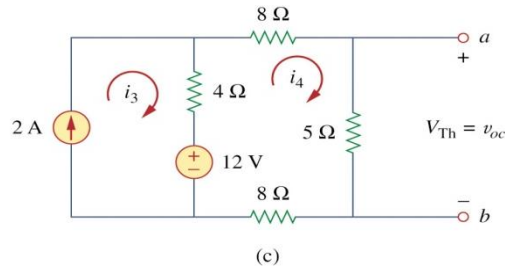
From equation (1) and (2), we get

$$i_2 = 1A = i_{sc} = I_N$$

Alternative method for I_N

$$I_N = \frac{V_{th}}{R_{th}}$$

V_{th} : open circuit voltage across terminals a and b in the following Figure.



- Mesh analysis

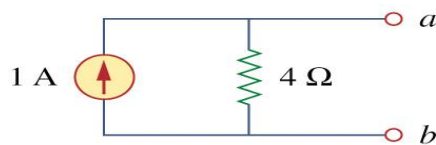
$$i_3 = 2 \text{ A}, \quad 25i_4 - 4i_3 - 12 = 0$$

$$\therefore i_4 = 0.8 \text{ A}$$

$$\therefore v_{oc} = V_{th} = 5i_4 = 4 \text{ V}$$

Hence,

$$I_N = \frac{V_{th}}{R_{th}} = \frac{4}{4} = 1 \text{ A}$$



Maximum Power Transfer:

Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen in the load ($R_L = R_{TH}$).

$$P_{max} = i^2 R_L = \left(\frac{V_{th}}{R_{th} + R_L} \right)^2 R_L = \frac{V_{th}^2}{4R_L}$$

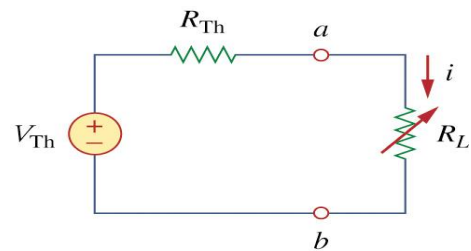


Figure: The circuit used for maximum power transfer.

OPERATIONAL AMPLIFIERS (OP AMPS):

- Op Amp is short for operational amplifier.
- An operational amplifier is modeled as a voltage controlled voltage source.
- An operational amplifier has very high input impedance and very high gain.

“Ideal” Op Amp

- The input resistance is infinite, $R_{in} = \infty$.
- The gain is infinite, $A = \infty$.
- Zero output resistance, $R_o = 0$.
- The op amp is in a negative feedback configuration.

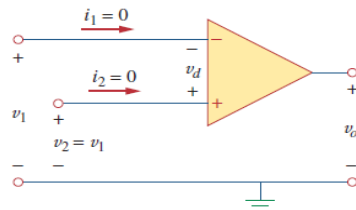


Figure: Ideal op amp model

Two Important characteristics of the ideal op amp are:

1. The currents into both input terminals are zero:

$$i_1 = 0, \quad i_2 = 0$$

2. The voltage across the input terminals is equal to zero:

$$v_d = v_2 - v_1 = 0$$

$$v_2 = v_1$$

Inverting Amplifier	Non-inverting Amplifier
<ul style="list-style-type: none"> • Non-inverting input terminal is connected to the ground. • Voltage source v_i connected to the inverting input terminal through R_i • Voltage gain, $\frac{v_o}{v_i} = -\frac{R_f}{R_i}$ 	<ul style="list-style-type: none"> • Voltage source v_i connected to directly at the non-inverting input terminal. • Resistor R_i is connected between the ground and the inverting terminal. • Voltage gain, $\frac{v_o}{v_i} = 1 + \frac{R_f}{R_i}$

Steps to solve an Op Amp problem:

Step 1: Apply KCL at one or both of the inputs.

Step 2: Apply two characteristic of the ideal op amp.

$$i_1 = 0, \quad i_2 = 0$$

$$v_2 = v_1 \text{ or } v_a = v_b$$

Step 3: Solve for the op amp output voltage.

Example:

Determine v_o of the following op amp circuit.

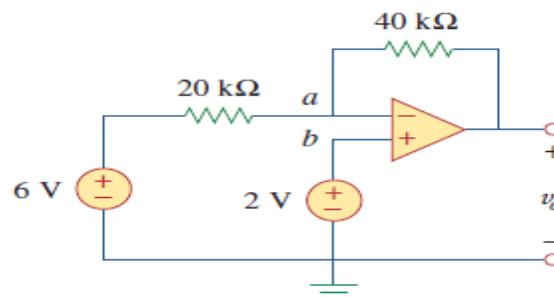


Figure: The op amp circuit.

Solution:

Step 1: Applying KCL at node a ,

$$\frac{v_a - v_b}{40 \times 1000} = \frac{6 - v_a}{20 \times 1000}$$

$$v_a - v_o = 12 - 2v_a \dots \dots \dots (1)$$

Step 2: For an ideal op amp,

$$v_a = v_b = 2 \text{ V} \dots \dots \dots (2)$$

Common Mistake

$$v_a = v_b = 0$$

Correct Statement

$$v_a = v_b = 2 \text{ V}$$

(2V is connected to the terminal b)

Step 3: From equation (1) and (2), we get

$$v_o = 3v_a - 12 = 3(2) - 12 = -6 \text{ V}$$

CAPACITORS AND INDUCTORS:**Capacitor:**

Definition: A capacitor consists of two conducting plates separated by an insulator (or dielectric). It can only store energy and can neither generate nor dissipate energy.

$$\text{Capacitor, } C = \frac{\epsilon A}{d}$$

$$\epsilon = \epsilon_r \epsilon_0, \quad \epsilon_0 = 8.854 \times 10^{-12} \left(\frac{F}{m} \right)$$

Feature:

Three factors affecting the value of capacitance:

1. **Area**: the larger the area, the greater the capacitance.
2. **Spacing between the plates**: the smaller the spacing, the greater the capacitance.
3. **Material permittivity**: the higher the permittivity, the greater the capacitance.

Circuit Symbols for Capacitors:

Figure: (a) Fixed Capacitor (b) Variable Capacitor

Charge in Capacitors:

$$q = cv$$

$$1F = 1C/V$$

Current in Capacitor, $i = C \frac{dv}{dt}$

Keep in mind the following things for the Capacitor:

- A capacitor is an open circuit to dc.

- The voltage on a capacitor cannot change abruptly.

Energy Storing in Capacitor:

$$w(t) = \frac{1}{2} C v^2(t) \quad w(t) = \frac{q^2(t)}{2C}$$

Examples:

1. Calculate the charge stored on a 3-pF capacitor with 20V across it.

Solution:

Since,
 $q = CV$

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

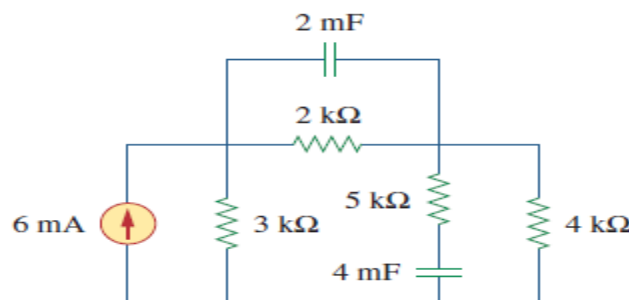
2. Find the energy stored in the capacitor.

Solution:

The energy stored is

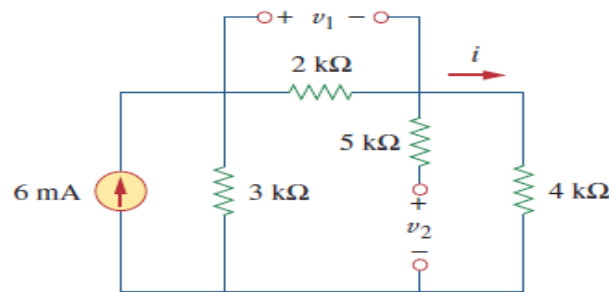
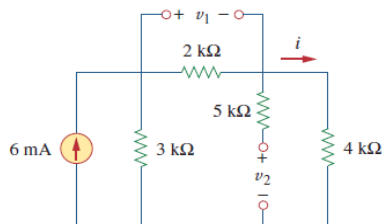
$$w = \frac{1}{2} C v^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

3. Find the energy stored in each capacitor in the following circuit under dc condition.



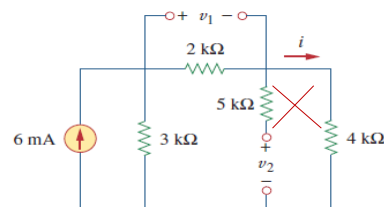
Solution:

Under dc condition, we replace each capacitor with an open circuit in the following Figure.

**Common Mistake**

By current division,

$$i = \frac{3}{3 + 2 + 4 + 5} (6 \text{ mA}) = 1.2 \text{ mA}$$

Correct Statement

$$i = \frac{3}{3 + 2 + 4} (6 \text{ mA}) = 2 \text{ mA}$$

By current division,

$$i = \frac{3}{3 + 2 + 4} (6 \text{ mA}) = 2 \text{ mA}$$

$$v_1 = 2(1000)(2 \times 10^{-3}) = 4 \text{ V}$$

$$v_2 = 4(1000)(2 \times 10^{-3}) = 8 \text{ V}$$

$$w_1 = \frac{1}{2} C_1 v_1^2 = \frac{1}{2} (2 \times 10^{-3}) (4)^2 = 16 \text{ mJ}$$

$$w_2 = \frac{1}{2} C_2 v_2^2 = \frac{1}{2} (4 \times 10^{-3}) (8)^2 = 128 \text{ mJ}$$

Inductor:

Definition: An inductor is made of a coil of conducting wire.

$$L = \frac{N^2 \mu A}{l}$$

$$\mu = \mu_r \mu_0$$

$$\mu_0 = 4\pi \times 10^{-7} \left(\frac{\text{H}}{\text{m}} \right)$$

N : Number of turns
 l : length
 A : cross – sectional area
 μ : permeability of the core

Voltage and Energy Stored in an Inductor:

$$\text{Voltage, } v = \frac{d\phi}{dt} = L \frac{di}{dt}$$

$$\text{Energy, } w(t) = \frac{1}{2} Li^2(t)$$

Keep in mind of the following things for an inductor.

- An inductor is like a [short circuit to dc](#).
- The current through an inductor cannot change instantaneously.

Example:

1. The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

Solution:

Since $v = L \frac{di}{dt}$ and $L = 0.1H$

$$\text{Voltage is } v = 0.1 \frac{d}{dt}(10te^{-5t}) = e^{-5t}(1 - 5t)V$$

The energy stored is

$$w(t) = \frac{1}{2} Li^2(t) = \frac{1}{2}(0.1)(100t^2e^{-10t}) = 5t^2e^{-10t}J$$

Summarize the most important characteristics of the three basic circuit elements:

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
$v - i$:	$v = iR$	$v = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0)$	$v = L \frac{di}{dt}$
$i - v$:	$i = \frac{v}{R}$	$i = C \frac{dv}{dt}$	$i = \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + i(t_0)$
p or w :	$p = i^2 R = \frac{v^2}{R}$	$w = \frac{1}{2} C v^2$	$w = \frac{1}{2} L i^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	not applicable	voltage (v)	current (i)

First Order Circuits

Definition: A first-order circuit is characterized by a first-order differential equation. It can only contain one energy storage element (a capacitor or an inductor). The circuit will also contain resistance.

Two types of first-order circuits:

- RC circuit
- RL circuit

Source-Free Circuits:

Definition: A **source-free circuit** is one where all independent sources have been disconnected from the circuit after some switch action. The voltages and currents in the circuit typically will have some transient response due to **initial conditions** (initial capacitor voltages and initial inductor currents).

Source-Free RC Circuits:Procedure:

Step 1: Apply KCL on the circuit

Step 2: Find the differential equation

Example:

Consider the RC circuit shown below.

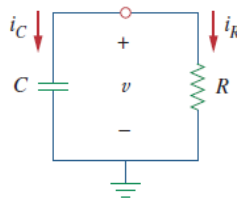


Figure: RC circuit

Note: It is source-free because no sources are connected to the circuit for $t > 0$.

Solution:

Step 1: Apply KCL to the top node of the above circuit.

$$i_C + i_R = 0$$

$$C \frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

Step 2: Solve the differential equation.

$$v(t) = V_0 e^{-t/RC}$$

V_0 = initial condition

$\tau = RC = \text{time constant}$

$$v(t) = V_0 e^{-t/\tau}$$

The Key to Working with a Source- Free RC Circuit Is Finding:

1. The initial voltage $v_{(0)} = V_0$ across the capacitor.
2. The time constant τ .

Source free RL Circuit:

Procedure:

Step 1: Apply KVL around the loop in the circuit.

Step 2: Find the differential equation.

Example:

Consider the RL circuit shown below.

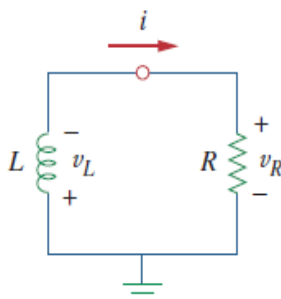


Figure: RL circuit

Step 1: Apply KVL around the loop,

$$v_L + v_R = 0$$

$$L \frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

Step 2: Solve the differential equation to show that,

$$i_{(t)} = I_0 e^{-\frac{Rt}{L}}$$

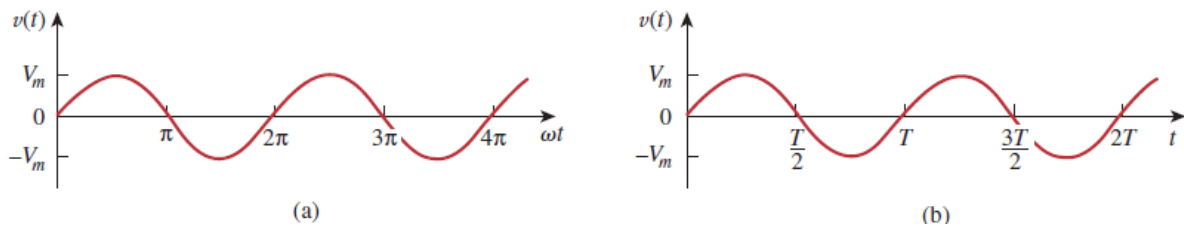
$$\tau = \frac{L}{R}$$

$$i_{(t)} = I_0 e^{-\frac{t}{\tau}}$$

AC CIRCUITS

SINUSOIDS AND PHASORSSinusoids:

- A sinusoid is a signal that has the form of the sine or cosine function.
- The sinusoidal current is referred to as an AC. Circuits driven by AC sources are referred to as AC Circuits
- Sketch of $V_m \sin \omega t$

Figure: (a) As a function of ωt . (b) As a function of t .

- V_m is the *amplitude* of the sinusoid
- ω is the *angular frequency* in radians/s
- ωt is the *argument* of the sinusoid
- f is the FREQUENCY in Hertz. $\omega = 2\pi f$ and $f = \frac{1}{T}$
- T is the period in seconds. $T = \frac{2\pi}{\omega}$

Example:

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12 \cos(50t + 10^\circ)$$

Solution:

The amplitude is: $V_m = 12$

The phase is: $\phi = 10^\circ$

The angular frequency is: $\omega = 50 \frac{\text{rad}}{\text{sec}}$.

The period is: $T = \frac{2\pi}{\omega} = \frac{2\pi}{50} = 0.1257 \text{ sec.}$

The frequency is: $f = \frac{1}{T} = \frac{1}{0.1257} = 7.958 \text{ Hz}$

Phase of Sinusoids:

Definition: A **periodic function** is one that satisfies $v(t) = v(t + nT)$, for all t and for all integers n .

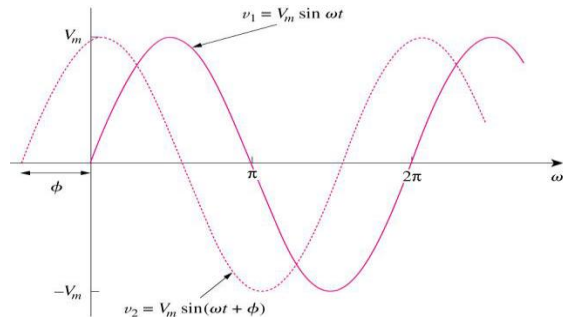
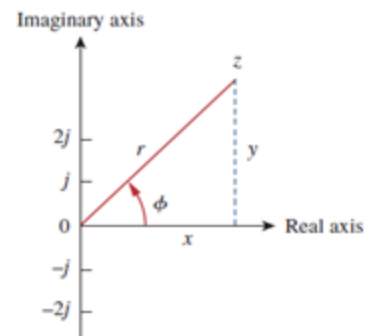


Figure: Two sinusoids with different phases.

- Only two sinusoidal values with the **same frequency** can be compared by their amplitude and phase difference.
- If the phase difference is zero, they are in phase.
- If the phase difference is not zero, they are out of phase.

Example :

Consider the sinusoidal voltage having phase ϕ ,
 $v(t) = V_m \sin(\omega t + \phi)$



Note that the following things in the above circuit:

- v_2 LEADS v_1 by phase ϕ
- v_1 LAGS v_2 by phase ϕ
- v_1 and v_2 are out of phase

Complex Numbers and Phasors:Complex Numbers:

Definition: A complex number may be written in the following form:

Rectangular form: $z = x + jy, j = \sqrt{-1}$

$$x = r \cos \theta, y = r \sin \theta$$

Polar form:

$$z = r \angle \phi$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}$$

Exponential form:

$$z = r e^{j\theta}$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}$$

Phasors:

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.
- Phasor is a mathematical equivalent of a sinusoid with time variable dropped.

$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m \angle \phi$
<div style="display: flex; justify-content: space-between;"> (Time Domain Re pr.) (Phasor Domain Representation) </div>
$v(t) = \text{Re}\{\mathbf{V} e^{j\omega t}\} \quad \text{(Converting Phasor back to time)}$

- Amplitude and phase difference are two principal concerns in the study of voltage and current sinusoids.
- Phasor will be defined from the cosine function in all our proceeding study. If a voltage or current expression is in the form of a sine, it will be changed to a cosine by subtracting from the phase.

Examples:

Transform the following sinusoids to phasors:

a. $i = 6 \cos(50t - 40^\circ)$

b. $v = -4 \sin(30t + 50^\circ)$

Solutions:

a. $\mathbf{I} = 6 \angle -40^\circ \text{ A}$

b. Since $-\sin(A) = \cos(A + 90^\circ)$;

$$v(t) = 4 \cos(30t + 50^\circ + 90^\circ) = 4 \cos(30t + 140^\circ) \text{ V}$$

Transform to phasor $\Rightarrow \mathbf{V} = 4 \angle 140^\circ \text{ V}$

Phasor Diagrams:

Sinusoid–phasor transformation:

Time Domain Representation	Phasor Domain Representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

Phasor Relationships for Circuit Elements:

Table: Summary of voltage current relationship of the circuit elements:

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L \frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C \frac{dv}{dt}$	$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$

Example:

If a voltage $v(t) = 100 \cos(60t + 20^\circ)$ V is applied to a parallel combination of a 40-k Ω resistor and a 50- μ F capacitor. Find the steady-state currents through the resistor and capacitor.

Solution:

Since R and C are in parallel, they have the same voltage across them. For the resistor,

$$V = I_R R \longrightarrow I_R = V / R = \frac{100 \angle 20^\circ}{40k} = 2.5 \angle 20^\circ \text{ mA}$$

$$i_R = \underline{2.5 \cos(60t + 20^\circ) \text{ mA}}$$

For the capacitor,

$$i_C = C \frac{dv}{dt} = 50 \times 10^{-6} (-60) \times 100 \sin(60t + 20^\circ) = \underline{-300 \sin(60t + 20^\circ) \text{ mA}}$$

Impedance and Admittance:

- The Impedance Z of a circuit is the ratio of phasor voltage V to the phasor current I .

$$Z = \frac{V}{I} \text{ or } V = ZI$$

- The Admittance Y of a circuit is the reciprocal of impedance measured in Simens (S).

$$Y = \frac{1}{Z}$$

Table: Impedances and admittances of passive elements.

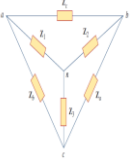
Element	Impedence	Admittance
R	$Z=R$	$Y = \frac{1}{R}$
L	$Z = j\omega L$	$Y = \frac{1}{j\omega L}$
C	$Z = \frac{1}{j\omega C}$	$Y = j\omega C$

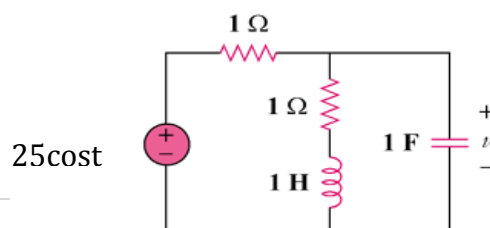
Note that $Z_L = j\omega L$ and $Z_c = \frac{1}{j\omega C}$.

Consider two extreme cases of angular frequency.

- When $\omega = 0$, $Z_L = 0$ (inductor is a short circuit at dc) and $Z_c = \infty$ (capacitor is an open circuit at dc)
- When $\omega = \infty$, $Z_L = \infty$ (inductor is an open circuit to high frequencies or AC ciircuit)and $Z_c = 0$

Some important formulas for AC and DC Circuits

Circuit	AC Circuits	DC Circuits
Series	Equivalent Impedance, $Z_{eq} = Z_1 + Z_2 + \dots Z_N$	Equivalent Resistance, $R_{eq} = R_1 + R_2 + \dots R_N$
Parallel	Equivalent Impedance, $\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \dots \frac{1}{Z_N}$	Equivalent Resistance, $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots \frac{1}{R_N}$
Current Division	$I_1 = \frac{Z_2}{Z_1 + Z_2} I, \quad I_2 = \frac{Z_1}{Z_1 + Z_2} I$	$I_1 = \frac{R_2}{R_1 + R_2} I, \quad I_2 = \frac{R_1}{R_1 + R_2} I$
Voltage Division	$V_1 = \frac{Z_1}{Z_1 + Z_2} V, \quad V_2 = \frac{Z_2}{Z_1 + Z_2} V$	$V_1 = \frac{R_1}{R_1 + R_2} V, \quad V_2 = \frac{R_2}{R_1 + R_2} V$
Conversion:		
Y- Δ	 $Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$ $Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$ $Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$	$R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$ $R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}$ $R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$
Δ -Y	$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$ $Z_2 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c}$ $Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$	$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$ $R_2 = \frac{R_a R_c}{R_a + R_b + R_c}$ $R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$

Example:Find $v(t)$ in the RLC circuit of the following Figure.

Solution:

$$\omega = 1,$$

$$1 \text{ H} \longrightarrow j\omega L = j(1)(1) = j$$

$$1 \text{ F} \longrightarrow \frac{1}{j\omega C} = \frac{1}{j(1)(1)} = -j$$

$$\mathbf{Z} = 1 + (1 + j) \parallel (-j) = 1 + \frac{-j+1}{1} = 2 - j$$

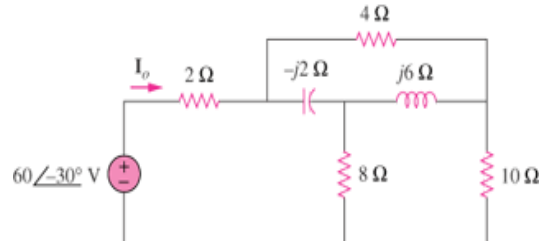
$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{25}{2 - j},$$

$$\mathbf{V}_c = \frac{(-j)(1 + j)}{1 + j - j} \mathbf{I} = (1 - j) \mathbf{I} = \frac{(1 - j)(25)}{2 - j} = 15.812 \angle -18.43^\circ$$

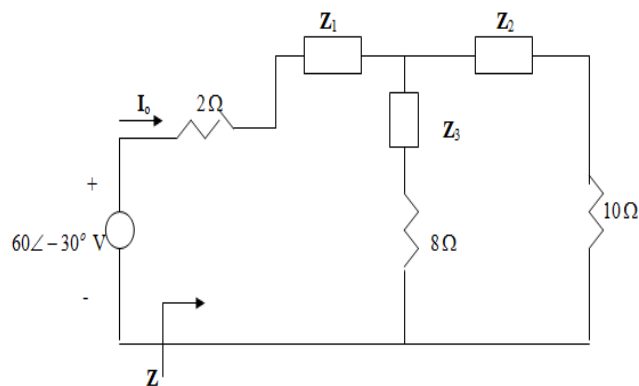
Thus, $v(t) = \mathbf{15.812 \cos(t - 18.43)}$

Example 2:

Find \mathbf{I}_0 in the following circuit.

Solution:

Step 1: Convert the delta to wye subnetwork as shown below.



$$Z_1 = \frac{-j2 \times 4}{4 + j4} = \frac{8 \angle -90^\circ}{5.6569 \angle 45^\circ} = -1 - j1, \quad Z_2 = \frac{j6 \times 4}{4 + j4} = 3 + j3,$$

$$Z_3 = \frac{12}{4 + j4} = 1.5 - j1.5$$

Step 2: Apply equivalent series and parallel impedances for the following elements.

$$(Z_3 + 8) // (Z_2 + 10) = (9.5 - j1.5) // (13 + j3) = 5.691 \angle 0.21^\circ = 5.691 + j0.02086$$

$$Z = 2 + Z_1 + 5.691 + j0.02086 = 6.691 - j0.9791$$

Step 3: solve for I_0 using ohm's law

$$I_0 = \frac{60 \angle -30^\circ}{Z} = \frac{60 \angle -30^\circ}{6.7623 \angle -8.33^\circ} = \underline{8.873 \angle -21.67^\circ \text{ A}}$$

SINUSOIDAL STEADY STATE ANALYSIS

Nodal and Mesh Analysis:

- Same procedure for solving AC circuit problems (see Method of analysis)

Superposition and Thevenin's and Norton theorem:

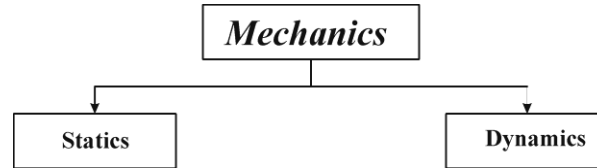
- Same procedure for solving AC circuit problem (see the Circuit Theorem)

REFERENCES:

Charles K. Alexander and Mathew N. O. Sadiku, Fundamental of Electric Circuits (Fifth Edition), McGraw Hill Higher Education

INTRODUCTION

Statics is a branch of Engineering Mechanics which is used to quantify forces and moments between bodies in static equilibrium. Force and moment are both vector quantities.



We can think of the equilibrium condition as applying to a) stationary bodies or b) bodies moving at constant velocity. In static equilibrium the sum of all resultant forces (1) and moments (2) are zero.

The following governing 3-D equations apply:

$$1. \quad \sum \mathbf{F} = \left(\sum F_x \right) \mathbf{i} + \left(\sum F_y \right) \mathbf{j} + \left(\sum F_z \right) \mathbf{k} = 0 \quad \text{or in scalar form}$$

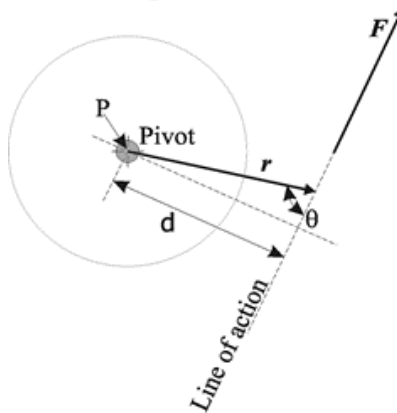
$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum F_z = 0$$

2. In vector form, the moment of a force \mathbf{F} about point P is defined as

$$\mathbf{M}_P = \mathbf{r} \times \mathbf{F}$$



The scalar moment magnitude is

$$\mathbf{M}_P = Fd$$

Units are in the form of N.m, kN.m, lb.ft, or lb.in.

The turning moment of a force can cause twisting or bending.

For the equilibrium of moments.....

$$\sum \mathbf{M} = \sum \mathbf{M}_x \mathbf{i} + \sum \mathbf{M}_y \mathbf{j} + \sum \mathbf{M}_z \mathbf{k} = 0 \quad \text{or in scalar form}$$

$$\sum M_x = 0$$

$$\sum M_y = 0$$

$$\sum M_z = 0$$

Please note:

- Multiple forces (vectors) in a system can be represented in terms of a resultant force
- Multiple moments (and or couples) in a system can also be represented as a resultant moment or couple.
- For equilibrium, the sum of all resultants (**R**), will be equal to zero.

The resultant of a force system is a force that will have the same effect on a body, if all the forces are removed and replaced by the resultant force only.

$$\mathbf{R} = R_x \mathbf{i} + R_y \mathbf{j} + R_z \mathbf{k}$$

Where

$$R_x = \sum F_x$$

$$R_y = \sum F_y$$

$$R_z = \sum F_z$$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

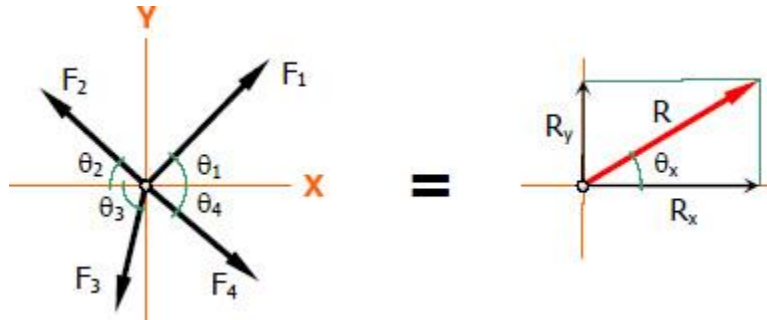
Direction Cosines (the angle the vector forms with each axis)

$$\cos \theta_x = \frac{R_x}{R}$$

$$\cos \theta_y = \frac{R_y}{R}$$

$$\cos \theta_z = \frac{R_z}{R}$$

Resultant of 2-D Concurrent Force System



$$R_x = \sum F_x$$

$$R_y = \sum F_y$$

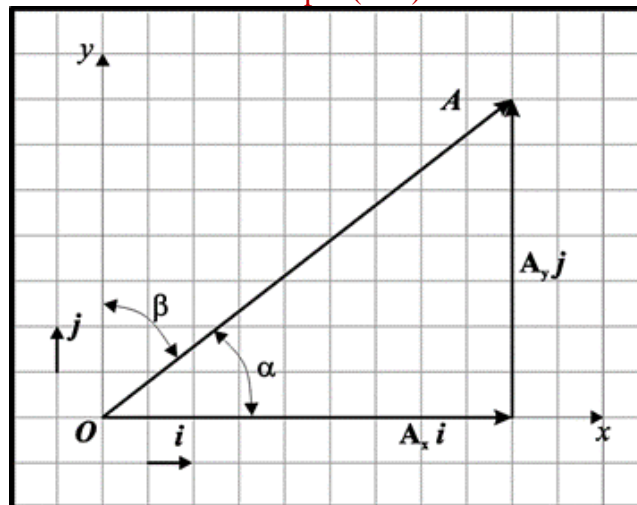
$$R = \sqrt{R_x^2 + R_y^2}$$

$$\tan \theta_x = \frac{R_y}{R_x}$$

RESOLVING VECTORS

A given vector A can be resolved into orthogonal components.

Example (2-D):



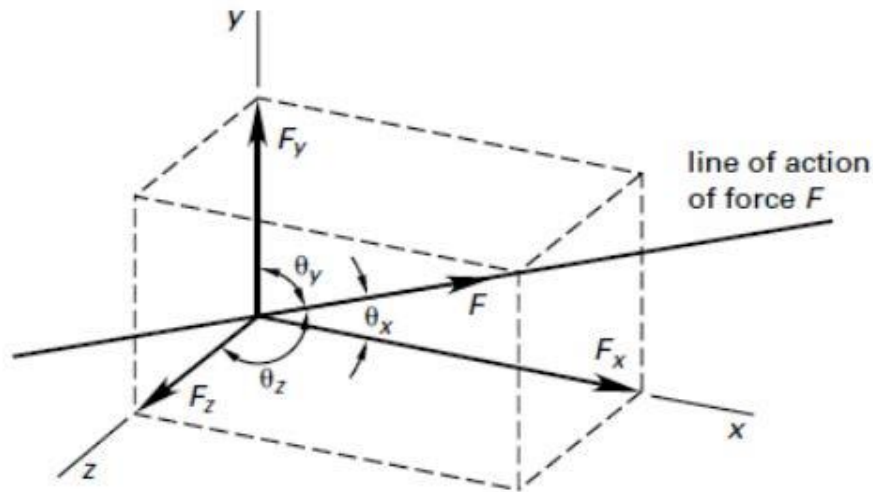
$$A = A_x i + A_y j$$

where

$$A_x = |A| \cos \alpha$$

$$A_y = |A| \cos \beta$$

Cos α and Cos β are called the direction cosines of vector A.

A force vector in 3-Dimensions:

$$F_x = F \cos \theta_x \quad F_y = F \cos \theta_y \quad F_z = F \cos \theta_z$$

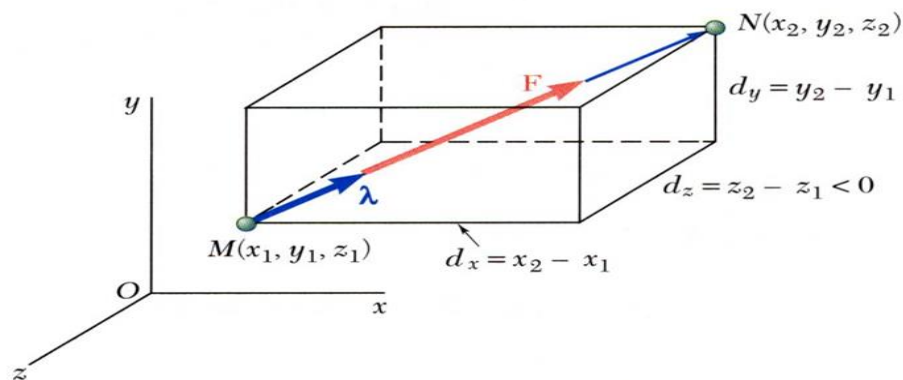
$$\begin{aligned} \vec{F} &= F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \\ &= F(\cos \theta_x \vec{i} + \cos \theta_y \vec{j} + \cos \theta_z \vec{k}) \\ &= F \vec{\lambda} \end{aligned}$$

$$\vec{\lambda} = \cos \theta_x \vec{i} + \cos \theta_y \vec{j} + \cos \theta_z \vec{k}$$

$\vec{\lambda}$ is the unit vector along the line of action of the vector \mathbf{F}

Rectangular components in space:

Given a vector \mathbf{F} defined by 2 points M and N in space.



\vec{d} = vector joining M and N

$$= d_x \vec{i} + d_y \vec{j} + d_z \vec{k}$$

$$d_x = x_2 - x_1 \quad d_y = y_2 - y_1 \quad d_z = z_2 - z_1$$

$$\vec{F} = F \vec{\lambda}$$

$$\vec{\lambda} = \frac{1}{d} (d_x \vec{i} + d_y \vec{j} + d_z \vec{k})$$

$$F_x = \frac{F d_x}{d} \quad F_y = \frac{F d_y}{d} \quad F_z = \frac{F d_z}{d}$$

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$$

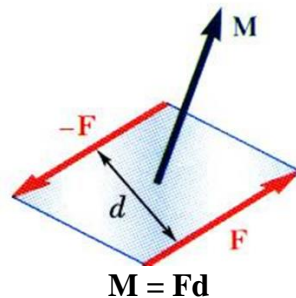
The magnitude (length) of the vector \mathbf{F} can be computed as

$$|\mathbf{F}| = \sqrt{\{F_x\}^2 + \{F_y\}^2 + \{F_z\}^2}$$

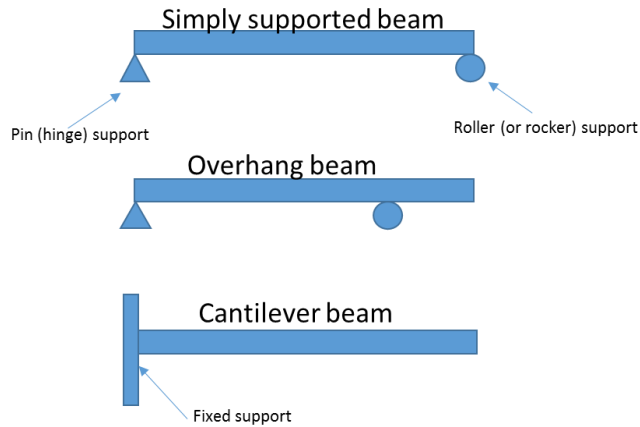
$$\vec{\lambda} = \mathbf{F}/|\mathbf{F}| = \text{Force vector/magnitude}$$

Couple

A couple is a system of forces that have the tendency to cause a body to rotate. A couple is represented by a pair of equal, opposite and parallel forces. The magnitude of the resultant force of a couple is zero. The magnitude M of the couple (magnitude of turning moment) is



A couple is a free vector that is independent of location which when applied at any point in a system produces the same effect.

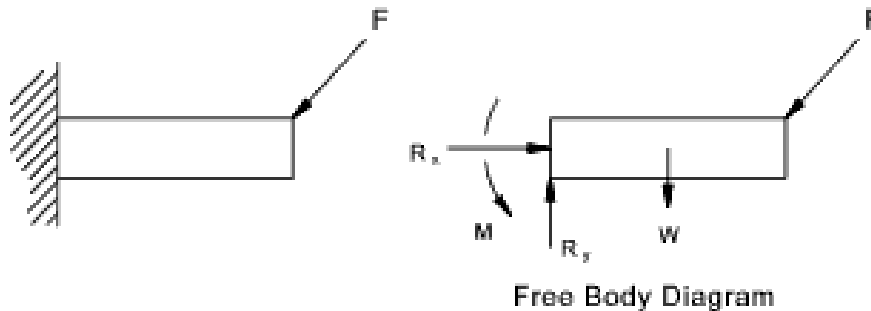


Common Statically Determinate Beams (above)

- Applied forces and or couples on a beam will result in reaction forces or moments at the supports.
- Different support types have different reactions.
- For beam equilibrium, the sum of all the forces and moments (or couples) including reactions must be equal to zero.

Example: 2-D Free Body Diagram of a cantilever beam with applied force F
<http://www.roymech.co.uk/>.

Note the moment reaction (M) and force (R_x & R_y) reactions at the fixed support end.



Centroids by Summation

The centroid refers to the geometric center of a region or object. It is the average point location of all the points in all of the coordinate directions. Calculating the centroid location is relevant in many engineering problems.

Centroids of lines

$$L \bar{x} = \sum lx$$

$$L \bar{y} = \sum ly$$

Centroids of Areas

$$A \bar{x} = \sum ax$$

$$A \bar{y} = \sum ay$$

Center of gravity of a homogeneous flat plate

$$W \bar{x} = \sum wx$$

$$W \bar{y} = \sum wy$$

Centroids of volumes

$$V \bar{x} = \sum vx$$

$$V \bar{y} = \sum vy$$

$$V \bar{z} = \sum vz$$

Center of gravity of bodies

$$W \bar{x} = \sum wx$$

$$W \bar{y} = \sum wy$$

$$W \bar{z} = \sum wz$$

Centroids by Integration:**Centroid of lines**

$$L \bar{x} = \int_a^b x_c dL$$

$$L \bar{y} = \int_a^b y_c dL$$

Centroids of Areas

$$A \bar{x} = \int_a^b x_c dA$$

$$A \bar{y} = \int_a^b y_c dA$$

$$\bar{x}A = \int x dA = Q_y$$

= first moment with respect to y

$$\bar{y}A = \int y dA = Q_x$$

= first moment with respect to x

Centroids of volumes

$$V \bar{x} = \int_a^b x_c dV$$

$$V \bar{y} = \int_a^b y_c dV$$

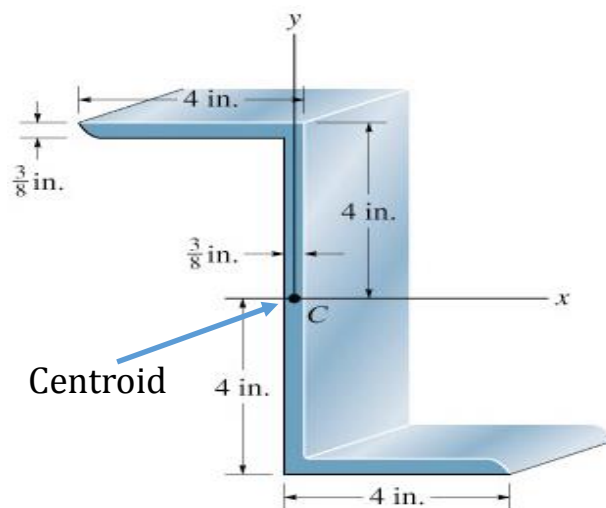
$$V \bar{z} = \int_a^b z_c dV$$

Center of gravity of bodies

$$W \bar{x} = \int_a^b x_c dW$$

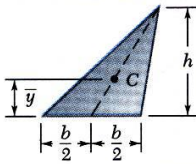
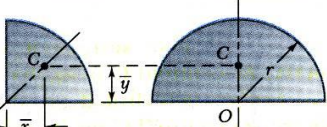

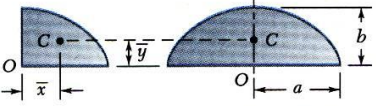

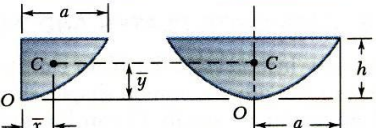
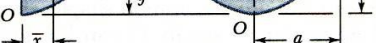
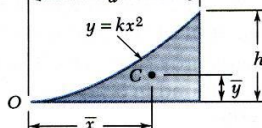
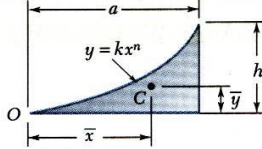
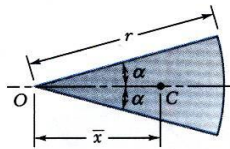
$$W \bar{y} = \int_a^b y_c dW$$

$$W \bar{z} = \int_a^b z_c dW$$



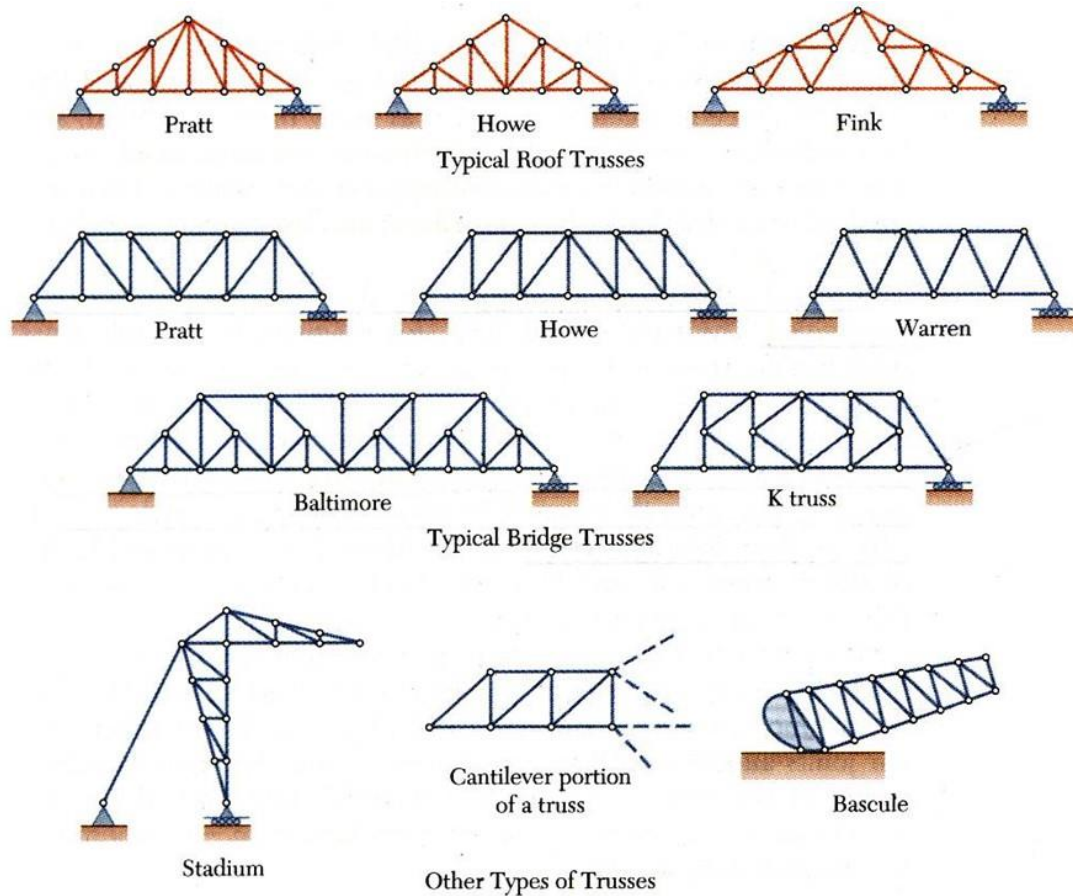
www.chegg.com

Centroids of common 2-D shapes

Shape		\bar{x}	\bar{y}	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2} a$	$\frac{n+1}{4n+2} h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

Analysis of Structures

Truss Structures: Truss structures are composed of structural members that carry axial loads. Truss members are connected by pinned joints and are assumed to be weightless. Truss structures create stability as a composite of triangular substructures with external loads applied only at the joints.

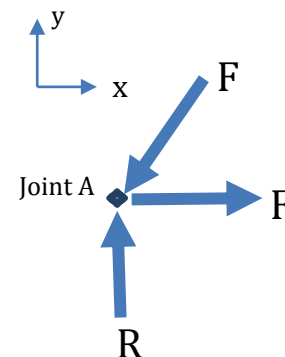
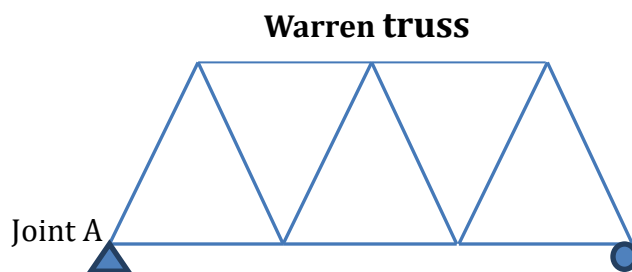


Methods to Analyze Truss Structures:

When we analyze trusses, the objective is to find the magnitude of the axial forces in each of the truss members. These axial forces are either tensile or compressive forces. Before using any method of analysis, first solve for the support reactions using the complete free body diagram (FBD) of the truss. Method of Joints and Method of Sections are outlined.

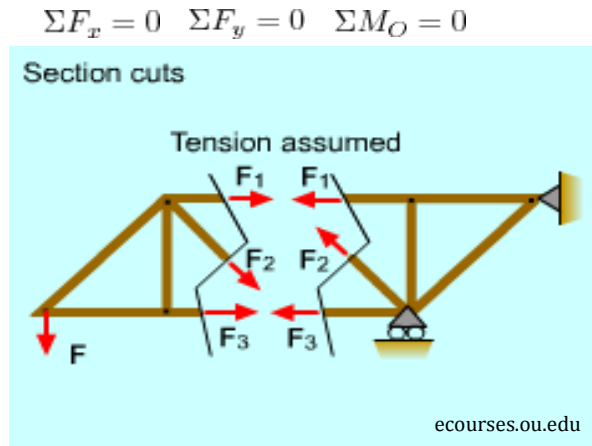
Method of Joints: The free-body diagram of each joint is a concurrent force system for which force equations apply

$$\Sigma F_x = 0 \quad \Sigma F_y = 0$$



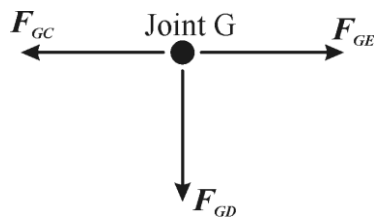
Apply equilibrium force equations to each Joint. Example free body diagram of joint A.

Method of Sections: Cut the truss into two sections using a cutting plane through the members whose internal forces we need to calculate. Each section may involve non-concurrent forces for which three equilibrium equations can be written.

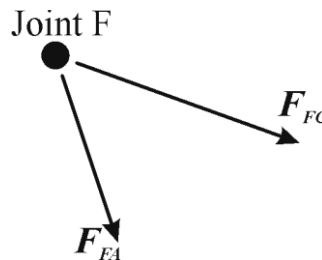


Zero Force Members: Under special conditions, certain truss members can have zero force.

Condition 1: At a joint where three members meet, if there is no external load and two of the members are collinear then the third non-collinear member is a zero-force member.



Condition 2: At a joint where two non-collinear members meet, if the joint is unloaded (no external force), both members are zero-force members.



Features of Zero Force Members:

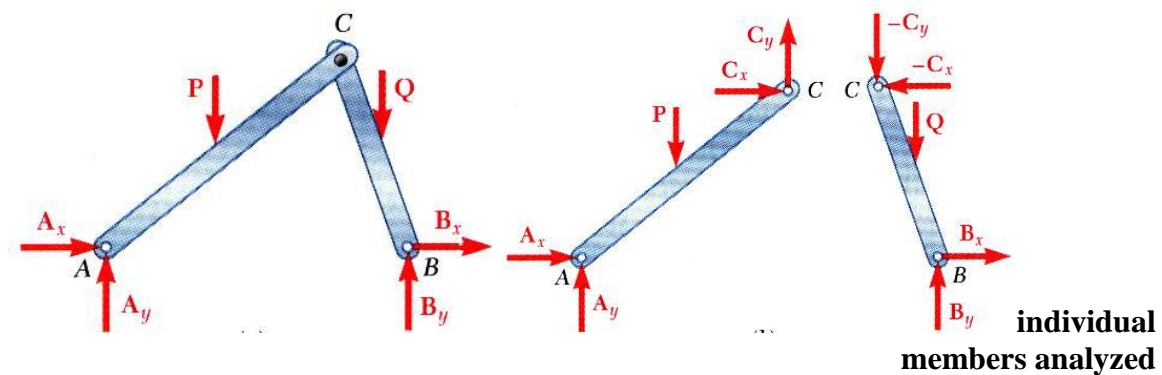
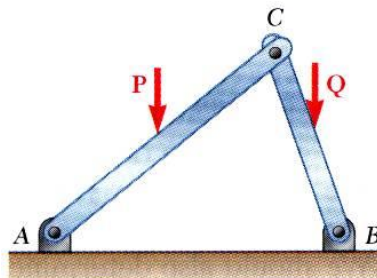
- These members provide stability of the structure by preventing buckling in long slender members under compressive forces.
- These members also carry load in the event that other members become overloaded.

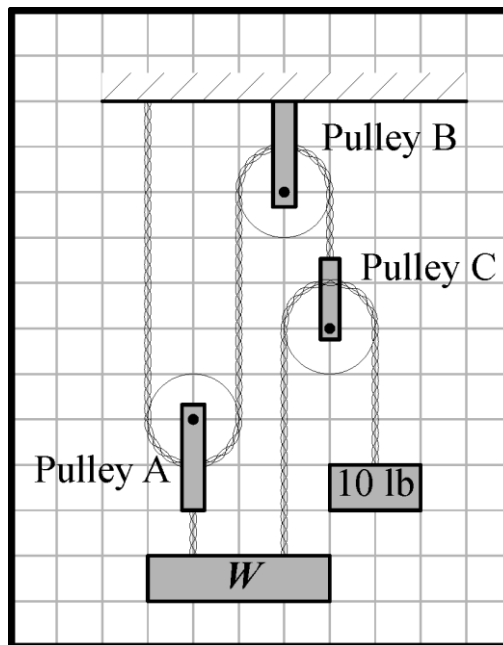
FRAMES AND MACHINES

Frames: Stationary structures that support more complex loads (like bending loads)

Machines: Complex systems that involve moving parts and provide mechanical advantage.

Solution to frame and machine problems include analysis of individual members that comprise the system.

Frames**Machines**



Friction:

Friction is the retarding force caused by the relative motion between two bodies in contact. Friction always acts opposite to the direction of the impending motion.

Two primary types of friction: Dry Friction (Coulomb friction) & Fluid Friction (lubrication applications). Here we deal with dry friction only.

Dry Friction:

Dry friction, occurs when unlubricated surfaces of two bodies are in contact and slide or tend to slide between each other.

Elements of Dry Friction

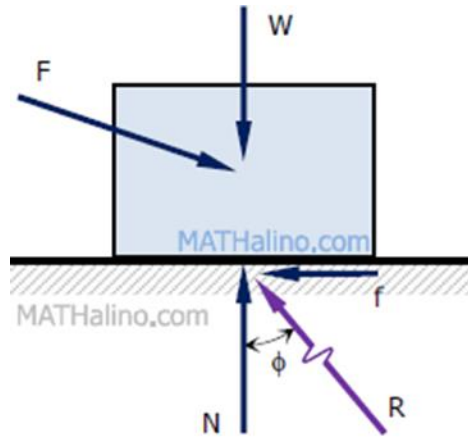
N = Total reaction perpendicular to the contact surface

f = Friction force

μ = Coefficient of friction

R = Resultant of f and N

ϕ = angle of friction

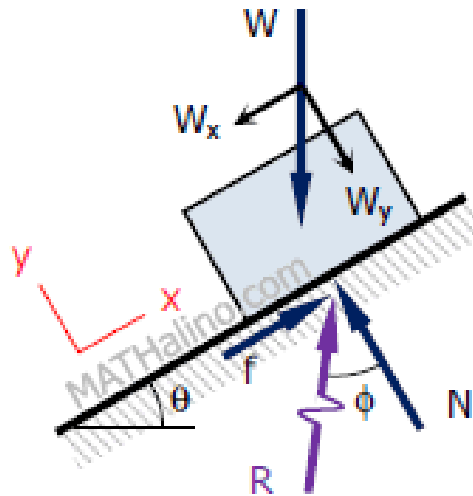


Formulas for Dry Friction (for impending motion)

$$f = \mu N$$

$$\tan \phi = \frac{f}{N}$$

$$\tan \phi = \mu$$



For the block shown, if it weighs W and is placed on an inclined plane at an angle θ with the horizontal.

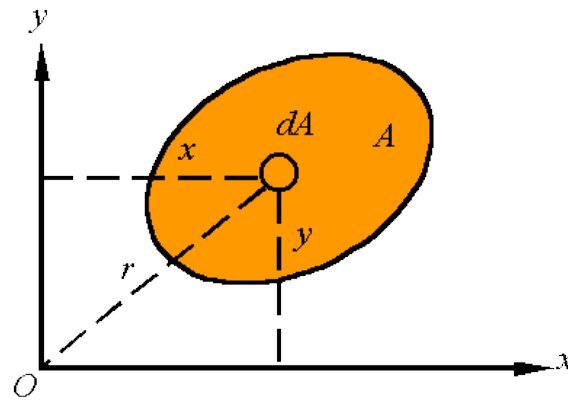
- If $\phi < \theta$ the friction force f is less than W_x . The block will slide down the plane.
- If $\phi = \theta$ the friction force f is equal to W_x . The block is in impending motion (about to slide down the plane).

- If $\phi > \theta$ frictional resistance f is greater than W_x the block remains stationary.

Please note : the maximum angle θ that a plane may be inclined without causing the body to slide down is equal to the angle of friction ϕ .

Moment of Inertia (MoI):

Moment of inertia is a geometric parameter called the second area moment. It is the product of area and the square of its moment arm about a reference axis



$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$

Moment of inertia about the x-axis I_x : Moment of inertia about the y-axis I_y

The polar area moment of inertia, J_O , is the area moment of inertia about the z-axis.

$$J_O = \int_A r^2 dA$$

since

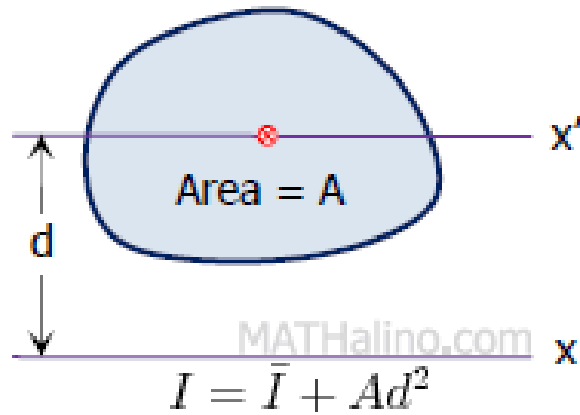
$$r^2 = x^2 + y^2$$

$$J_O = I_x + I_y$$

The radius of gyration is the distance k away from an axis that all the area can be concentrated to result in the same moment of inertia.

$$I = k^2 A$$

Parallel Axis Theorem



where

x' = centroidal axis

x = any axis parallel to the centroidal axis

I = moment of inertia about the x -axis

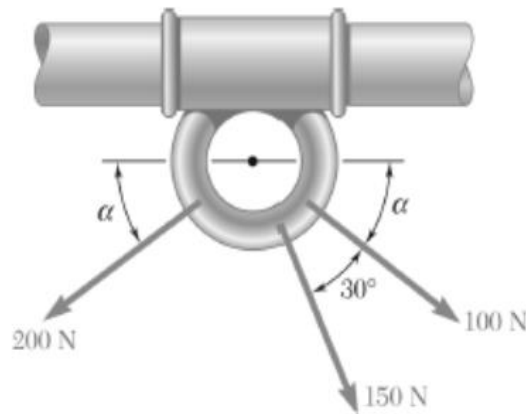
\bar{I} = centroidal moment of inertia

A = area of the section

d = distance between x and x'

APPENDIX A - Selected Examples:1. Resolving forces, resultant forces

Knowing that $\alpha = 35^\circ$, determine the resultant of the three forces shown.



100-N Force:

$$F_x = +(100 \text{ N}) \cos 35^\circ = +81.915 \text{ N}$$

$$F_y = -(100 \text{ N}) \sin 35^\circ = -57.358 \text{ N}$$

150-N Force:

$$F_x = +(150 \text{ N}) \cos 65^\circ = +63.393 \text{ N}$$

$$F_y = -(150 \text{ N}) \sin 65^\circ = -135.946 \text{ N}$$

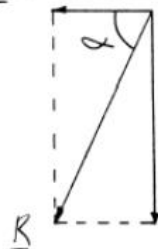
200-N Force:

$$F_x = -(200 \text{ N}) \cos 35^\circ = -163.830 \text{ N}$$

$$F_y = -(200 \text{ N}) \sin 35^\circ = -114.715 \text{ N}$$

Force	x Comp. (N)	y Comp. (N)
100 N	+81.915	-57.358
150 N	+63.393	-135.946
200 N	-163.830	-114.715
	$R_x = -18.522$	$R_y = -308.02$

$$R_x = -18.522 \text{ j}$$



$$R_y = -308.02 \text{ j}$$

$$\mathbf{R} = R_x \mathbf{i} + R_y \mathbf{j}$$

$$= (-18.522 \text{ N}) \mathbf{i} + (-308.02 \text{ N}) \mathbf{j}$$

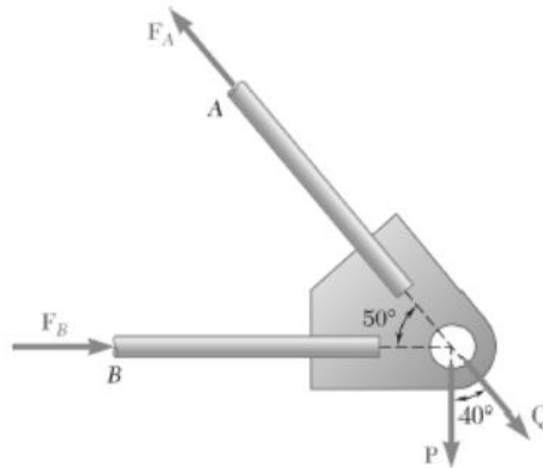
$$\tan \alpha = \frac{R_y}{R_x}$$

$$= \frac{308.02}{18.522}$$

$$\alpha = 86.559^\circ$$

$$R = \frac{308.02 \text{ N}}{\sin 86.559^\circ}$$

$$\mathbf{R} = 309 \text{ N} \nearrow 86.6^\circ$$

2. Equilibrium in a concurrent force system

Two forces **P** and **Q** are applied as shown to an aircraft connection. Knowing that the connection is in equilibrium and that $P = 500 \text{ lb}$ and $Q = 650 \text{ lb}$, determine the magnitudes of the forces exerted on the rods **A** and **B**.

Resolving the forces into *x*- and *y*-directions:

$$\mathbf{R} = \mathbf{P} + \mathbf{Q} + \mathbf{F}_A + \mathbf{F}_B = 0$$

Substituting components:

$$\begin{aligned} \mathbf{R} = & -(500 \text{ lb})\mathbf{j} + [(650 \text{ lb}) \cos 50^\circ]\mathbf{i} \\ & - [(650 \text{ lb}) \sin 50^\circ]\mathbf{j} \\ & + F_B\mathbf{i} - (F_A \cos 50^\circ)\mathbf{i} + (F_A \sin 50^\circ)\mathbf{j} = 0 \end{aligned}$$

In the *y*-direction (one unknown force):

$$-500 \text{ lb} - (650 \text{ lb}) \sin 50^\circ + F_A \sin 50^\circ = 0$$

Thus,

$$F_A = \frac{500 \text{ lb} + (650 \text{ lb}) \sin 50^\circ}{\sin 50^\circ}$$

$$= 1302.70 \text{ lb}$$

$$F_A = 1303 \text{ lb} \quad \blacktriangleleft$$

In the *x*-direction:

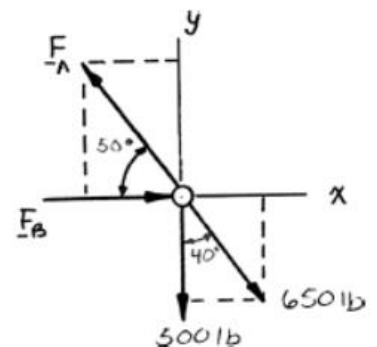
$$(650 \text{ lb}) \cos 50^\circ + F_B - F_A \cos 50^\circ = 0$$

Thus,

$$\begin{aligned} F_B &= F_A \cos 50^\circ - (650 \text{ lb}) \cos 50^\circ \\ &= (1302.70 \text{ lb}) \cos 50^\circ - (650 \text{ lb}) \cos 50^\circ \\ &= 419.55 \text{ lb} \end{aligned}$$

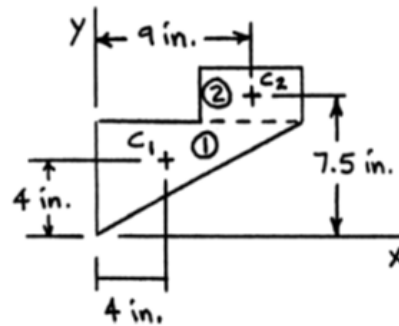
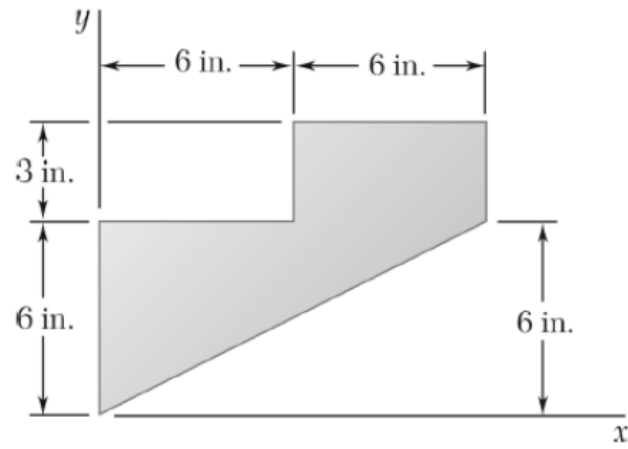
$$F_B = 420 \text{ lb} \quad \blacktriangleleft$$

Free-Body Diagram



3. Centroid of areas

Locate the centroid of the plane area shown.



	A, in^2	\bar{x}, in	\bar{y}, in	$\bar{x}A, \text{in}^3$	$\bar{y}A, \text{in}^3$
1	$\frac{1}{2}(12)(6) = 36$	4	4	144	144
2	$(6)(3) = 18$	9	7.5	162	135
Σ	54			306	279

Then

$$\bar{X}A = \Sigma \bar{x}A$$

$$\bar{X}(54) = 306$$

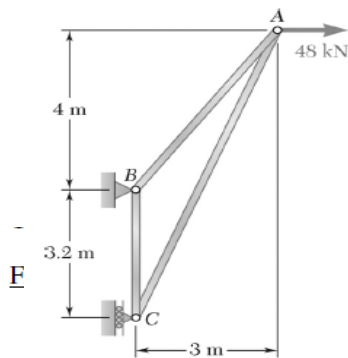
$$\bar{X} = 5.67 \text{ in.}$$

$$\bar{Y}A = \Sigma \bar{y}A$$

$$\bar{Y}(54) = 279$$

$$\bar{Y} = 5.17 \text{ in.}$$

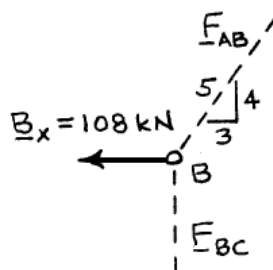
4. Structural Analysis – Method of Joints



Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

$$\begin{aligned}
 &= 0 \quad B_y = 0 \\
 &+\circlearrowleft \Sigma M_C = 0: -B_x(3.2 \text{ m}) - (48 \text{ kN})(7.2 \text{ m}) = 0 \\
 &B_x = -108 \text{ kN} \quad B_x = 108 \text{ kN} \leftarrow \\
 &+\rightarrow \Sigma F_x = 0: C - 108 \text{ kN} + 48 \text{ kN} = 0 \\
 &C = 60 \text{ kN} \quad C = 60 \text{ kN} \rightarrow
 \end{aligned}$$

Free body: Joint B:

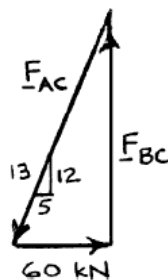
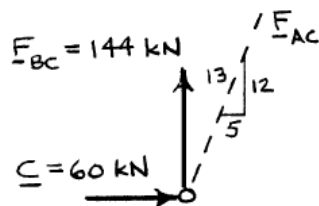


$$\frac{F_{AB}}{5} = \frac{F_{BC}}{4} = \frac{108 \text{ kN}}{3}$$

$$F_{AB} = 180.0 \text{ kN} \quad T \blacktriangleleft$$

$$F_{BC} = 144.0 \text{ kN} \quad T \blacktriangleleft$$

Free body: Joint C:



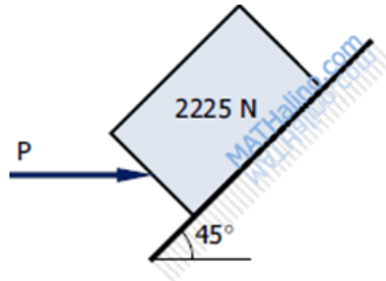
$$\frac{F_{AC}}{13} = \frac{F_{BC}}{12} = \frac{60 \text{ kN}}{5}$$

$$F_{AC} = 156.0 \text{ kN} \quad C \blacktriangleleft$$

$$F_{BC} = 144 \text{ kN} \quad (\text{checks})$$

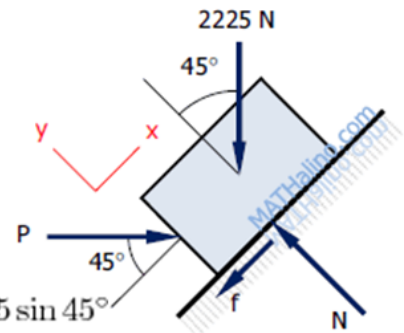
5. Friction

The 2225-N block shown is in contact with 45° incline. The coefficient of static friction is 0.25. Compute the value of the horizontal force P necessary to (a) just start the block up the incline or (b) just prevent motion down the incline.



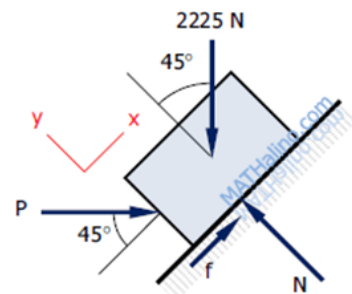
Part (a) – Force P to just start the block to move up the incline

$$\begin{aligned}\Sigma F_y &= 0 \\ N &= 2225 \cos 45^\circ + P \sin 45^\circ \\ N &= 1573.31 + 0.7071P \\ f &= \mu N = 0.25(1573.31 + 0.7071P) \\ f &= 393.33 + 0.1768P \\ \Sigma F_x &= 0 \\ P \cos 45^\circ &= f + 2225 \sin 45^\circ \\ P \cos 45^\circ &= (393.33 + 0.1768P) + 2225 \sin 45^\circ \\ 0.5303P &= 1966.64 \\ P &= 3708.55 \text{ N}\end{aligned}$$



Part (b) – Force P to just prevent the block to slide down the incline

$$\begin{aligned}\Sigma F_y &= 0 \\ N &= 2225 \cos 45^\circ + P \sin 45^\circ \\ N &= 1573.31 + 0.7071P \\ f &= \mu N = 0.25(1573.31 + 0.7071P) \\ f &= 393.33 + 0.1768P \\ \Sigma F_x &= 0 \\ P \cos 45^\circ + f &= 2225 \sin 45^\circ \\ P \cos 45^\circ + (393.33 + 0.1768P) &= 2225 \sin 45^\circ \\ 0.8839P &= 1179.98 \\ P &= 1335 \text{ N}\end{aligned}$$



REFERENCES

- Vector Mechanics for Engineers: Statics and Dynamics, Beer/Johnston 10th Ed. (course textbook)
- Engineering Statics by Sridhar Condoor
- Mathalino.com
- emweb.unl.edu/NEGAHBAN/EM223/note18/note18.htm
- oli.cmu.edu/ (Open Learning Initiative)

1. THERMODYNAMICS LAWS

Zeroth law of thermodynamics: If two bodies are in thermal equilibrium with a third body, they are also in thermal equilibrium with each other.

First law of thermodynamics: The change in a system's internal energy is equal to the difference between heat added to the system from its surroundings and work done by the system on its surroundings.

Second law of thermodynamics: It is impossible for a process to have as its sole result the transfer of heat from a cooler body to a hotter one.

2. PRESSURE CHANGE

Absolute pressure: The actual pressure at a given position. It is measured relative to absolute vacuum (i.e., absolute zero pressure).

Gage pressure: The difference between the absolute pressure and the local atmospheric pressure. Most pressure-measuring devices are calibrated to read zero in the atmosphere, and so they indicate gage pressure.

Vacuum pressures: Pressures below atmospheric pressure.

$$P_{\text{gage}} = P_{\text{abs}} - P_{\text{atm}}$$

P_{abs} = actual (absolute) pressure

$$\frac{dp}{dz} = -\rho g$$

$$\Delta P = P_2 - P_1 = -\int_1^2 (\rho g) dz$$

Example 1:

The pressure in a pressurized water tank is measured by a multi-fluid manometer. *The gage pressure of air in the tank is to be determined.*

The densities of mercury, water, and oil are given to be 13,600, 1000, and 850 kg/m³, respectively. Consider $h_1=0.2\text{m}$, $h_2=0.3\text{m}$ and $h_3=0.46$

Solution:

The pressure in a pressurized water tank is measured by a multi-fluid manometer. The gage pressure of air in the tank is to be determined.

Assumptions The air pressure in the tank is uniform (i.e., its variation with elevation is negligible due to its low density), and thus we can determine the pressure at the air-water interface.

Properties The densities of mercury, water, and oil are given to be 13,600, 1000, and 850 kg/m³, respectively.

Analysis Starting with the pressure at point 1 at the air-water interface, and moving along the tube by adding (as we go down) or subtracting (as we go up) the ρgh terms until we reach point 2, and setting the result equal to P_{atm} since the tube is open to the atmosphere gives

$$P_1 + \rho_{\text{water}} gh_1 + \rho_{\text{oil}} gh_2 - \rho_{\text{mercury}} gh_3 = P_{\text{atm}}$$

Solving for P_1 ,

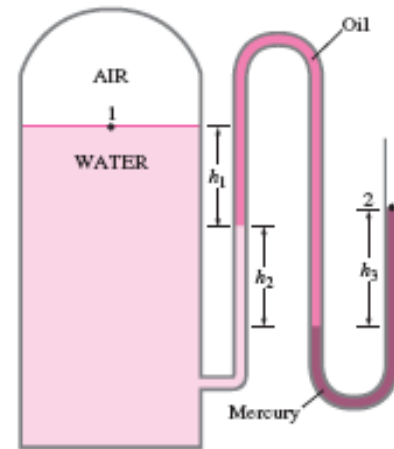
$$P_1 = P_{\text{atm}} - \rho_{\text{water}} gh_1 - \rho_{\text{oil}} gh_2 + \rho_{\text{mercury}} gh_3$$

or,

$$P_1 - P_{\text{atm}} = g(\rho_{\text{mercury}} h_3 - \rho_{\text{water}} h_1 - \rho_{\text{oil}} h_2)$$

Noting that $P_{1,\text{gage}} = P_1 - P_{\text{atm}}$ and substituting,

$$\begin{aligned} P_{1,\text{gage}} &= (9.81 \text{ m/s}^2)[(13,600 \text{ kg/m}^3)(0.46 \text{ m}) - (1000 \text{ kg/m}^3)(0.2 \text{ m}) \\ &\quad - (850 \text{ kg/m}^3)(0.3 \text{ m})] \left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2} \right) \left(\frac{1 \text{ kPa}}{1000 \text{ N/m}^2} \right) \\ &= 56.9 \text{ kPa} \end{aligned}$$



3. READING THE TABLES

How do I find properties of steam or refrigerant 134a?

When finding data for steam or refrigerant 134a, always begin your search in the saturation tables. When the temperature is given, use the saturation temperature table; and if the pressure is given, use the saturation pressure table. Then compare the other data that you have, such as specific volume, specific internal energy, specific enthalpy, or specific entropy, to the saturation values (v_f and v_g , u_f and u_g , etc.). Now, ask the three questions as follows (Here, specific volume is used as the example; but, the same technique applies to the other properties u , h , and s):

(a) Is $v < v_f$ at the state? If so, the state is in the compressed liquid region. Approximate all the intensive data, v , u , h , and s , as the saturated liquid value at the given value of the temperature. Do not use the saturation data at the given pressure because this would result in a very poor approximation to compressed liquid data.

(b) Is $v_f < v < v_g$ at the state? If so, the state is in the two-phase or saturation region. Use the given value of the specific volume to determine the quality at the state. This is done by solving the mass-weighted averaged equation, $v = v_f + x(v_g - v_f)$, for the quality x . Then use the quality x to determine the other intensive properties by a similar mass-weighted average equation for those properties. The pressure and temperature for the state are those equal to the saturation values.

(c) Is $v_g < v$ at the state? If so, the state is in the superheat region. Go to the superheat table and look for the state. You may have to interpolate to get an answer.

If you are given the pressure, find the saturation temperature, T_{sat} , from the saturation pressure table or estimate the saturation temperature at the pressure from the saturation temperature table. If the given temperature is greater than the T_{sat} , the state is superheated, and use the superheat tables to determine the other data. If the temperature is less than the T_{sat} , the state is compressed, and approximate all the intensive data, v , u , h , and s , as the saturated liquid value at the temperature, not the pressure.

If you are given the temperature, get the P_{sat} from the saturation table. If the given P is greater than P_{sat} , the state is compressed. Approximate all the intensive data, v , u , h , and s , as the saturated liquid value at the temperature, not the pressure.

You may look for wording that says the state is a saturated liquid. If so, that state lies on the saturated liquid line, and use v_f , u_f , h_f , and s_f as the intensive values. Or, if the state is a saturated vapor, the state lies on the saturated vapor line, and use v_g , u_g , h_g , and s_g as the intensive values.

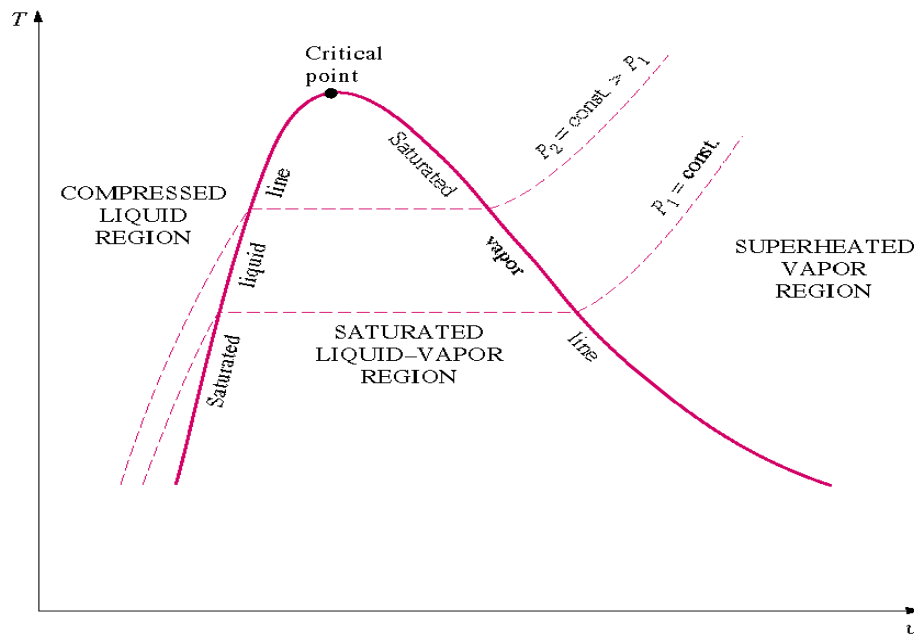


Figure: T-V diagram

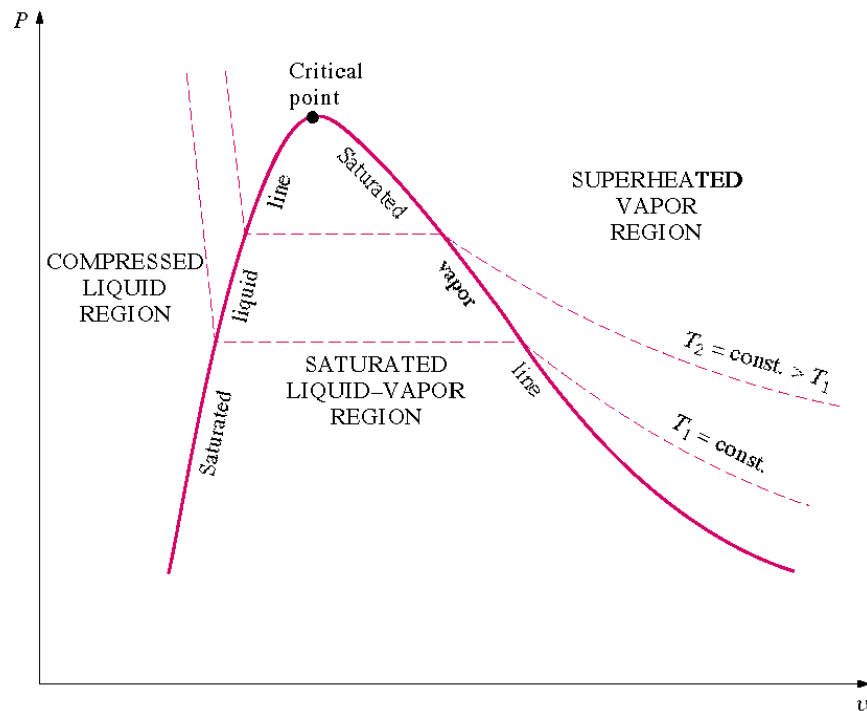


Figure: P-V diagram

Example 2:

Complete the table below for water

$T, ^\circ\text{C}$	P, kPa	$v, \text{m}^3/\text{kg}$	Phase description
50		7.72	
	400		Saturated vapor
250	500		
110	350		

Solution:

To complete the above table, you refer to given parameters and compare them to critical values.

$T, ^\circ\text{C}$	P, kPa	$v, \text{m}^3/\text{kg}$	Phase description
50	12.35	7.72	Saturated mixture
143.6	400	0.4624	Saturated vapor
250	500	0.4744	Superheated vapor

110	350	0.001051	<i>Compressed liquid</i>
-----	-----	----------	--------------------------

4. QUALITY

Quality x is the ratio of the mass of vapor to the total mass of a saturated mixture. The quality lies in the range $0 \leq x \leq 1$.

Saturated liquid–vapor mixture is a mixture of the liquid and vapor phases that coexist in equilibrium.

Any extensive property "y" of a saturated mixture is given by:

$$y = y_f + x (y_g - y_f)$$

Example 3:

A piston-cylinder device that is filled with R-134a is cooled at constant pressure. The final temperature and the change of total internal energy are to be determined if the final volume is half the initial volume.

Solution:

Analysis The initial specific volume is

$$\nu_1 = \frac{V}{m} = \frac{12.322 \text{ m}^3}{100 \text{ kg}} = 0.12322 \text{ m}^3/\text{kg}$$

The initial state is superheated and the internal energy at this state is

$$\left. \begin{array}{l} P_1 = 200 \text{ kPa} \\ \nu_1 = 0.12322 \text{ m}^3/\text{kg} \end{array} \right\} u_1 = 263.08 \text{ kJ/kg (Table A-13)}$$

The final specific volume is

$$\nu_2 = \frac{\nu_1}{2} = \frac{0.12322 \text{ m}^3/\text{kg}}{2} = 0.06161 \text{ m}^3/\text{kg}$$

This is a constant pressure process. The final state is determined to be saturated mixture whose temperature is

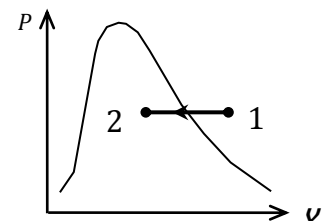
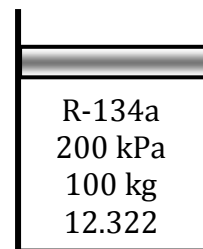
$$T_2 = T_{\text{sat @ 200 kPa}} = -10.09^\circ\text{C (Table A-12)}$$

The internal energy at the final state is (Table A-12)

$$x_2 = \frac{\nu_2 - \nu_f}{\nu_{fg}} = \frac{(0.06161 - 0.0007533) \text{ m}^3/\text{kg}}{(0.099867 - 0.0007533) \text{ m}^3/\text{kg}} = 0.6140$$

$$u_2 = u_f + x_2 u_{fg} = 38.28 + (0.6140)(186.21) = 152.61 \text{ kJ/kg}$$

Hence, the change in the internal energy is



$$\Delta u = u_2 - u_1 = 152.61 - 263.08 = -110.47 \text{ kJ/kg}$$

5. HEAT ENGINE

Heat engines are cyclic devices that convert heat into work; an example is a steam power plant.

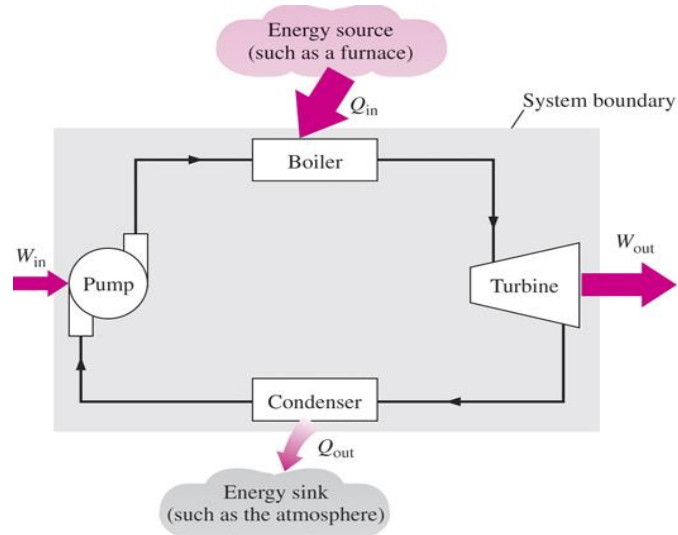


Figure: Heat Engine

Q_{in} = amount of heat supplied to steam in boiler from a high-temperature source (furnace)

Q_{out} = amount of heat rejected from steam in condenser to a low-temperature sink (the atmosphere, a river, etc.)

W_{out} = amount of work delivered by steam as it expands in turbine

W_{in} = amount of work required to compress water to boiler pressure

In the above diagram, the net work is equal to:

$$W_{net,out} = W_{out} - W_{in}$$

$$W_{net,out} = Q_{in} - Q_{out}$$

The efficiency of a heat engine is given by:

$$\eta = \frac{W_{out}}{Q_{in}}$$

Example 4:

A power plant produces 600MW at 40% efficiency. Find the rate of heat rejected to the surrounding.

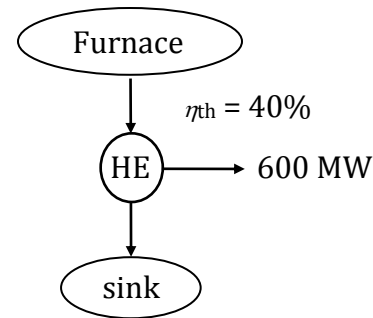
Solution:

Analysis The rate of heat supply to the power plant is determined from the thermal efficiency relation,

$$\dot{Q}_H = \frac{\dot{W}_{\text{net,out}}}{\eta_{\text{th}}} = \frac{600 \text{ MW}}{0.4} = 1500 \text{ MW}$$

The rate of heat transfer to the river water is determined from the first law relation for a heat engine,

$$\dot{Q}_L = \dot{Q}_H - \dot{W}_{\text{net,out}} = 1500 - 600 = \mathbf{900 \text{ MW}}$$

**6. HEAT PUMPS**

Heat pumps are cyclic devices that transfer heat from a low-temperature medium to a high-temperature one.

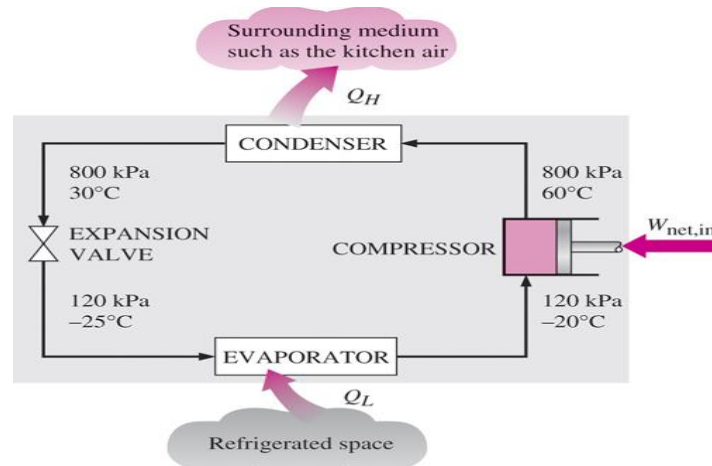


Figure: Refrigerator

The *efficiency* of a refrigerator is expressed in terms of the **coefficient of performance (COP)**.

$$COP_R = \frac{\text{Desired Output}}{\text{Required Input}} = \frac{Q_L}{W_{\text{net}}} = \frac{Q_L}{Q_H - Q_L}$$

Example 5:

A refrigerator operates on the ideal vapor compression refrigeration cycle with R-134a as the working fluid between the pressure limits of 120 kPa and 800 kPa at a rate of 18 grams per second. If the compressor power is 1.2kW, calculate the COP of the heat pump and the rate of heat absorbed from the outside air

Solution:

Refrigerant-134a flows through the condenser of a residential heat pump unit. For a given compressor power consumption the COP of the heat pump and the rate of heat absorbed from the outside air are to be determined.

Assumptions 1 The heat pump operates steadily. 2 The kinetic and potential energy changes are zero.

Properties The enthalpies of R-134a at the condenser inlet and exit are

$$\left. \begin{array}{l} P_1 = 800 \text{ kPa} \\ T_1 = 35^\circ\text{C} \end{array} \right\} h_1 = 271.22 \text{ kJ/kg}$$

$$\left. \begin{array}{l} P_2 = 800 \text{ kPa} \\ x_2 = 0 \end{array} \right\} h_2 = 95.47 \text{ kJ/kg}$$

Analysis (a) An energy balance on the condenser gives the heat rejected in the condenser

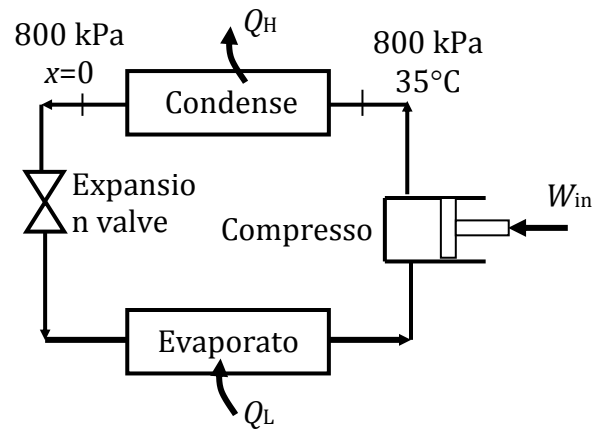
$$\dot{Q}_H = \dot{m}(h_1 - h_2) = (0.018 \text{ kg/s})(271.22 - 95.47) \text{ kJ/kg} = 3.164 \text{ kW}$$

The COP of the heat pump is

$$\text{COP} = \frac{\dot{Q}_H}{\dot{W}_{\text{in}}} = \frac{3.164 \text{ kW}}{1.2 \text{ kW}} = \mathbf{2.64}$$

(b) The rate of heat absorbed from the outside air

$$\dot{Q}_L = \dot{Q}_H - \dot{W}_{\text{in}} = 3.164 - 1.2 = \mathbf{1.96 \text{ kW}}$$

**7. CARNOT CYCLE**

Carnot cycle is a totally reversible cycle. Therefore, all the processes that comprise it can be reversed.

In any Carnot cycle, the heat transfer ratio Q_H/Q_L is the same as the absolute temperature ratio T_H/T_L .

There exist Carnot heat engines, Carnot refrigerators and Carnot heat pumps. Their efficiencies are:

$$\text{Heat Engine } \eta_{th,rev} = 1 - \frac{T_L}{T_R}$$

$$\text{Refrigerator } COP_{R,rev} = \frac{1}{\frac{T_H}{T_L} - 1}$$

$$\text{Heat Pump } COP_{HP,rev} = \frac{1}{1 - \frac{T_L}{T_H}}$$

8. OTTO AND DIESEL CYCLES

Otto cycles refer to spark ignition engines and Diesel cycles refer to compression ignition engines.

The efficiency of the Otto cycle is a function of the compression ratio, r , and specific heat ratio. The efficiency of a diesel cycle is a function of the compression ratio, r , cut-off ratio, r_c , and specific heat ratio k .

$$\eta_{th,otto} = \frac{w_{net}}{q_{in}} = 1 - \frac{1}{r^{k-1}}$$

$$\eta_{th,diesel} = \frac{w_{net}}{q_{in}} = 1 - \frac{1}{r^{k-1}} \left[\frac{r_c^k - 1}{k(r_c - 1)} \right]$$

Example 6:

An ideal Otto cycle produces 90kW of power and has a compression ratio of 10.5. Find the thermal efficiency and the rate of heat input.

Solution:

Assumptions 1 The air-standard assumptions are applicable. 2 Kinetic and potential energy changes are negligible. 3 Air is an ideal gas with constant specific heats.

Properties The properties of air at room temperature are $c_p = 1.005 \text{ kJ/kg}\cdot\text{K}$, $c_v = 0.718 \text{ kJ/kg}\cdot\text{K}$, and $k = 1.4$ (Table A-2a).

Analysis The definition of cycle thermal efficiency reduces to

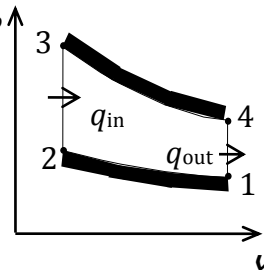
$$\eta_{th} = 1 - \frac{1}{r^{k-1}} = 1 - \frac{1}{10.5^{1.4-1}} = 0.6096 = \mathbf{61.0\%}$$

The rate of heat addition is then

$$\dot{Q}_{in} = \frac{\dot{W}_{net}}{\eta_{th}} = \frac{90 \text{ kW}}{0.6096} = \mathbf{148 \text{ kW}}$$

Example 7:

An ideal diesel cycle has a compression ratio of 20 and a cutoff ratio of 1.3. If the initial temperature in the cylinder is 15°C , find the maximum temperature of the air and the rate of heat addition.



Solution:

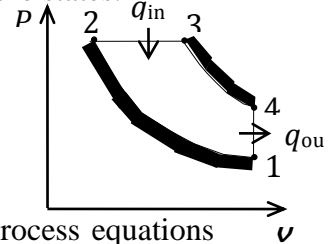
Assumptions **1** The air-standard assumptions are applicable. **2** Kinetic and potential energy changes are negligible. **3** Air is an ideal gas with constant specific heats.

Properties The properties of air at room temperature are $c_p = 1.005 \text{ kJ/kg}\cdot\text{K}$, $c_v = 0.718 \text{ kJ/kg}\cdot\text{K}$, $R = 0.287 \text{ kJ/kg}\cdot\text{K}$, and $k = 1.4$ (Table A-2a).

Analysis We begin by using the process types to fix the temperatures of the states.

$$T_2 = T_1 \left(\frac{v_1}{v_2} \right)^{k-1} = T_1 r^{k-1} = (288 \text{ K})(20)^{1.4-1} = 954.6 \text{ K}$$

$$T_3 = T_2 \left(\frac{v_3}{v_2} \right) = T_2 r_c = (954.6 \text{ K})(1.3) = \mathbf{1241 \text{ K}}$$



Combining the first law as applied to the various processes with the process equations gives

$$\eta_{\text{th}} = 1 - \frac{1}{r^{k-1}} \frac{r_c^k - 1}{k(r_c - 1)} = 1 - \frac{1}{20^{1.4-1}} \frac{1.3^{1.4} - 1}{1.4(1.3 - 1)} = 0.6812$$

According to the definition of the thermal efficiency,

$$\dot{Q}_{\text{in}} = \frac{\dot{W}_{\text{net}}}{\eta_{\text{th}}} = \frac{250 \text{ kW}}{0.6812} = \mathbf{367 \text{ kW}}$$

9. AIR CONDITIONING

The psychrometric chart presents moist air properties in a convenient form. They are used extensively in A-C applications. The psychrometric chart serves as a valuable aid in visualizing the A-C processes such as heating, cooling, and humidification.

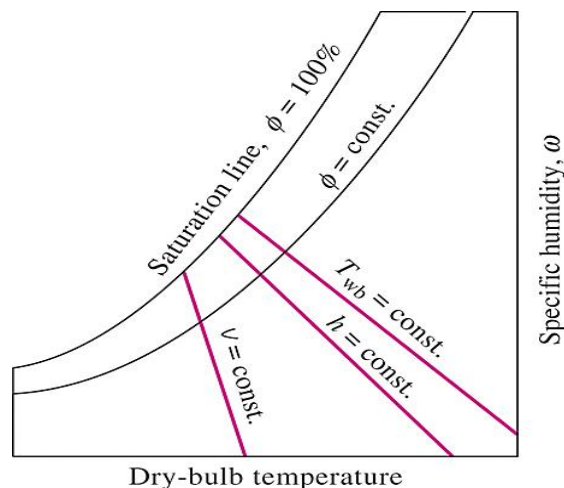


Figure: Schematic for a psychrometric chart.

Maintaining a given area at the desired temperature and humidity requires air-conditioning processes.

These processes include *simple heating* (raising the temperature), *simple cooling* (lowering the temperature), *humidifying* (adding moisture), and *dehumidifying* (removing moisture).

Sometimes a combination of two of these processes is needed to bring the air to a desired temperature and humidity level; air is commonly heated and humidified in winter and cooled and dehumidified in summer.

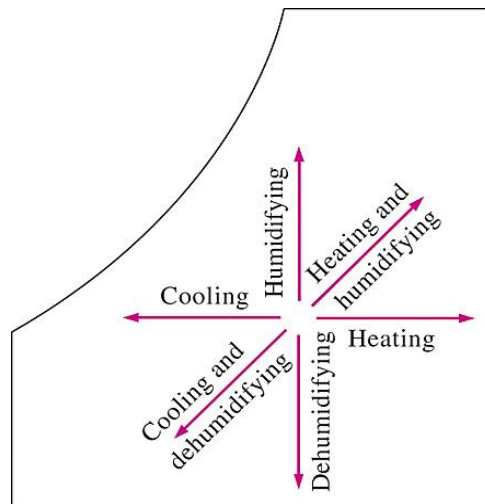


Figure: Air-conditioning processes

Example 8:

Air enters an evaporative cooler at 1 atm, 36°C, and 20 percent relative humidity at a rate of 4 m³/min, and it leaves with a relative humidity of 90 percent. Determine

- The exit temperature of the air
- The required rate of water supply to the evaporative cooler

Analysis (a) From the psychrometric chart, at 36°C and 20% relative humidity we read

$$T_{wb1} = 19.5^\circ\text{C}$$

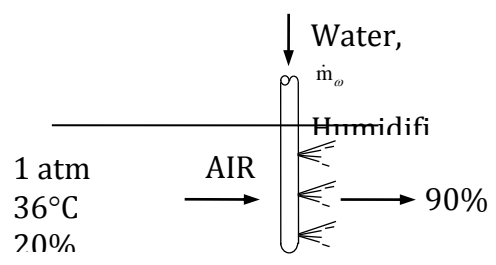
$$\omega_1 = 0.0074 \text{ kg H}_2\text{O/kg dry air}$$

$$\nu_1 = 0.887 \text{ m}^3/\text{kg dry air}$$

Assuming the liquid water is supplied at a temperature not much different than the exit temperature of the air stream, the evaporative cooling process follows a line of constant wet-bulb temperature. That is,

$$T_{wb2} \cong T_{wb1} = 19.5^\circ\text{C}$$

At this wet-bulb temperature and 90% relative humidity we read



$$T_2 = 20.5^\circ \text{C}$$

$$\omega_2 = 0.0137 \text{ kg H}_2\text{O} / \text{kg dry air}$$

Thus air will be cooled to 20.5°C in this evaporative cooler.

(b) The mass flow rate of dry air is

$$\dot{m}_a = \frac{\dot{V}_1}{\nu_1} = \frac{4 \text{ m}^3 / \text{min}}{0.887 \text{ m}^3 / \text{kg dry air}} = 4.51 \text{ kg/min}$$

Then the required rate of water supply to the evaporative cooler is determined from

$$\begin{aligned} \dot{m}_{\text{supply}} &= \dot{m}_{w2} - \dot{m}_{w1} = \dot{m}_a (\omega_2 - \omega_1) \\ &= (4.51 \text{ kg/min})(0.0137 - 0.0074) \\ &= \mathbf{0.028 \text{ kg/min}} \end{aligned}$$

10. COMMON MISTAKES TO AVOID

1. **Units**—Equations must be dimensionally sound. The failure to use units in a consistent manor is one of the most frequent errors made in the solution to thermodynamics problems. One must recognize that every term in an equation must have the same set of units. Often the knowledge of the units assigned to the parameters of a problem will lead to the definition of an equation that applies to the problem solution.
2. **System definition (closed vs. open)**—Systems must be well defined before a solution can proceed. Using energy balance equations developed for the open system or control volume cannot led to a successful completion of the solution to a closed system problem. Using energy balance equations developed for the closed system cannot led to a successful completion of the solution to an open system problem.
3. **Algebra**—Simple algebraic mistakes often occur in the solutions to thermodynamics problems. These mistakes unnecessarily lead to a great deal of student frustration that is not related to the thermodynamics theory applied to the problem solution.
4. **System sketches**—Students often omit sketches of the system hardware and/or omit indications of energy interactions at the system boundaries. These sketches are invaluable to the thought process that must take place to understand the requirements of the problem solution. When sketching systems, show the directions of mass transfer, if any, across the system boundaries and the direction of any heat transfer or work crossing the system boundary.
5. **Specific heats**—Often there is confusion over the concepts associated with specific heats. The question asked by students is often “Do I use c_p or c_v in this solution?” The answer to this question depends on whether one needs to determine enthalpy changes or internal energy changes of the working fluid. If the working fluid of the problem is an ideal gas, c_p is used to calculated changes in enthalpy and c_v is used to calculate changes in internal energy. Because the enthalpy and internal energy of ideal gases are functions of temperature only, the methods for determining enthalpy

changes, $\Delta h = \Delta c_p(T)dT$, and internal energy changes, $\Delta u = \Delta c_v(T)dT$, are true for an ideal gas in any process.

6. **Interpreting processes**—Often the importance of the role that the process plays in the solution technique applied to a thermodynamic problem is ignored. A process describes how the properties of a thermodynamic system are related as the process takes place. Most of the processes that are used in thermodynamics are ones in which one thermodynamic property is held constant. We must visualize a mechanical means in which the property is held constant. For instance, an isothermal (constant temperature) process can be accomplished by placing the system in an oil bath in which the oil temperature is controlled. As the process within the system takes place, the system's temperature is maintained constant. This requires that the other properties, pressure, volume, internal energy, enthalpy, entropy, etc., must all change together in a way to maintain the temperature constant. As a consequence of the process there may be heat and work energies exchanged between the system and its surroundings. For a closed system, the boundary work can be determined from $w_b = \int P(v) dv$, and the process relation is used to determine pressure as a function of volume so that the integral can be completed.
7. **Solution technique**—One of the most important aspects of developing a solution to a thermodynamic problem is to follow a systemic solution technique.
8. **Integrals of PdV** —Boundary work done by a working fluid within a system is given as the integral of PdV . Before one can successfully perform this integral, the process of the working fluid within the system must be known and the pressure as a function of volume must be determined. This integral is not, in general, equal to $P_2V_2 - P_1V_1$. This result is correct only when the process is constant pressure, and $P = P_1 = P_2$.
9. **Temperature**—When the value of temperature is substituted into an equation such as the ideal equation of state, the temperature must be in absolute units. Tabular data, such as the steam tables, are frequently tabulated as functions of temperature in Celsius or Fahrenheit units. This technique is often used to tabulate data because the absolute values of temperature are obtained by adding the appropriate constant to the ordinary temperature for each system of units. A temperature change in the ordinary units, Celsius or Fahrenheit, is the same as the temperature change measured in Kelvin or rankine units.
10. **Pressure**—When the value of pressure is substituted into an equation such as the ideal equation of state, the pressure must in absolute units. The absolute pressure is calculated as atmospheric pressure plus/minus gage pressure. Since atmospheric pressure may vary daily or even hourly, tabular data, such as the steam tables, are always tabulated as functions of absolute pressure.
11. **Isothermal process versus adiabatic process**—An isothermal process is a process in which the temperature is maintained constant, and an adiabatic process is one in which there is no heat transfer. To maintain the temperature of a system constant

during a process there may be heat transfer between the system and its surroundings where the surroundings will have a temperature greater than or less than that of the system. An isothermal process is not adiabatic. We often get confused with this concept because heat transfer is energy in transition due to a temperature difference. In this case the temperature difference for heat transfer occurs between the constant temperature system and its surroundings.

1. LIMIT

A. If the function is a polynomial, rational function, power function, trigonometric function, inverse trigonometric function, exponential function or logarithmic function, and a is in the domain, then $\lim_{x \rightarrow a} f(x) = f(a)$.

For example:

$$\lim_{x \rightarrow 1} x^2 - 3x + 4 = 2, \lim_{x \rightarrow 1} \frac{x^2 + 1}{x + 1} = 1, \lim_{x \rightarrow 4} \sqrt{x} = 2, \lim_{x \rightarrow \pi} \sin x = 0$$

B. Properties of limits: Suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$ and

$\lim_{x \rightarrow a} g(x)$ exist. Then:

a. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$ For example:

$$\lim_{x \rightarrow \pi} [x + \sin x - \cos x] = \lim_{x \rightarrow \pi} x - \lim_{x \rightarrow \pi} \sin x + \lim_{x \rightarrow \pi} \cos x = \pi - 1$$

b. $\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$. For example, $\lim_{x \rightarrow 0} 5 \cos x = 5 \lim_{x \rightarrow 0} \cos x = 5$.

c. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$. For example.

$$\lim_{x \rightarrow 0} (x^2 + 1) \cos x = \lim_{x \rightarrow 0} (x^2 + 1) \lim_{x \rightarrow 0} \cos x = 1$$

d. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$. For example: $\lim_{x \rightarrow 3} \frac{x^2}{2^x} = \frac{\lim_{x \rightarrow 3} x^2}{\lim_{x \rightarrow 3} 2^x} = \frac{9}{8}$.

C. L'Hospital Rule. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$.

a. $\frac{0}{0}$ -type: For example: $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x} = \lim_{x \rightarrow 0} \frac{e^x}{\cos x} = 1$

b. $\frac{\infty}{\infty}$ -type: For example: $\lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$.

c. $0 \cdot \infty$ -type: write $f \cdot g = \frac{f}{1/g}$ or $\frac{g}{1/f}$. For example.

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0.$$

d. 0^0 -, 1^∞ - and ∞^0 -type. For example: Find the limit $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x$. To find the

1^∞ -type limit, we take logarithm first, we have $\ln(1 + \frac{1}{x})^x = x \ln(1 + \frac{1}{x})$.

Moreover, by L'Hospital's rule, we have

$$\lim_{x \rightarrow \infty} x \ln(1 + \frac{1}{x}) = \lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{x}} \cdot (-\frac{1}{x^2})}{-\frac{1}{x^2}} = 1. \text{ Then, following } e^x \text{ is a}$$

continuous function, we have $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = \lim_{x \rightarrow \infty} e^{x \ln(1 + \frac{1}{x})} = e^{\lim_{x \rightarrow \infty} x \ln(1 + \frac{1}{x})} = e$.

2. DERIVATIVES.

A. Basic formula.

a. Power, Exponential and Logarithmic Functions.

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(a^x) = a^x \ln a$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

b. Trigonometric functions

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$

c. Inverse Trigonometric functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
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$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$	$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$
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B. Differential Rules.

- $[cf(x)]' = cf'(x)$. For example: $[2 \tan x]' = 2 \sec^2 x$.
- $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$. For example: $[x^3 - 2x^2 + 4x]' = 3x^2 - 4x + 4$.
- $[f(x) \cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$ [Product Rule]. For example.
 $[x^2 \sin x]' = [x^2]' \sin x + x^2 [\sin x]' = 2x \sin x + x^2 \cos x$.
 Common Error: $[x^2 \sin x]' = [x^2]' \cdot [\sin x]' = 2x \cos x$.
- $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'g - fg'}{g^2}$ [Quotient Rule]. For example
 $\left[\frac{x^2}{\tan x}\right]' = \frac{2x \tan x - x^2 \sec^2 x}{\tan^2 x}$.
- $[f(g(x))]' = f'(g(x)) \cdot g'(x)$ [Chain Rule]. The crucial part of the chain rule is to identify the inner function and outer function. For example,
 $[\sin^2 x]' = 2 \sin x \cos x$ and $[\sin(x^2)]' = \cos(x^2) \cdot 2x$.

C. Partial Derivative is differentiation with respect to a single variable, with all other variables being treated as constants. For example, consider the function

$$f(x, y) = x^2 - 2xy + \sin y, \text{ then } \frac{\partial f}{\partial x} = 2x - 2y \text{ and } \frac{\partial f}{\partial y} = -2x + \cos y.$$

D. Implicit Differentiation. For example, find $\frac{dy}{dx}$ for the equation

$$x \cdot \sin y + y \cdot \cos x = 0.$$

Method 1: Manipulating this equation to find y as a function of x is out of the question. So, we use implicit differentiation as follows:

Differentiate this equation respect x using the product rule and chain rule:

$$\frac{d}{dx}(x) \sin y + x \frac{d}{dx}(\sin y) + \frac{d}{dx}(y) + y \frac{d}{dx}(\cos x) = 0 \text{ implying that}$$

$$\sin y + x \cos y \frac{dy}{dx} + \cos x \frac{dy}{dx} - y \sin x = 0. \text{ Solving for } \frac{dy}{dx}, \text{ we have}$$

$$\frac{dy}{dx} = \frac{y \sin x - \sin y}{x \cos y + \cos x}.$$

Method 2: We can also find $\frac{dy}{dx}$ by partial derivative. Let $f(x, y) = x \sin y + y \cos x$

then
$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{\sin y - y \sin x}{x \cos y + \cos x} = \frac{y \sin x - \sin y}{x \cos y + \cos x}.$$

3. INDEFINITE INTEGRAL

Indefinite integral $\int f(x) dx$ is the general anti-derivative of $f(x)$.

A. Basic Formula.

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$	$\int \frac{dx}{x} = \ln x + c$	$\int a^x dx = \frac{a^x}{\ln a} + c$
$\int e^x dx = e^x + c$	$\int \sin x dx = -\cos x + c$	$\int \cos x dx = \sin x + c$
$\int \sec^2 x dx = \tan x + c$	$\int \csc^2 x dx = -\cot x + c$	$\int \sec x \tan x dx = \sec x + c$
$\int \frac{dx}{\sqrt{x^2 - a^2}} = \sin^{-1} \frac{x}{a} + c$	$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$	$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

B. U-substitution. $\int f(g(x)) g'(x) dx = \int f(u) du$ where $u = g(x)$. For example, find the indefinite integral $\int x^3 \sqrt{2x^4 - 1} dx$.

Let $u = 2x^4 - 1$, then $du = 8x^3 dx$. This implies that

$$\int x^3 \sqrt{2x^4 - 1} dx = \int \frac{1}{8} \sqrt{u} du = \frac{1}{12} u^{3/2} + c = \frac{1}{12} (2x^4 - 1)^{3/2} + c.$$

C. Integration by parts. $\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$ or $\int u dv = uv - \int v du$. Choose u and dv based on the mnemonic LIATE which stands for "Logarithmic, Inverse(trigonometric), Algebraic, Trigonometric, Exponential". The u function should be whatever appears first in the list. For example, find the indefinite integral $\int x \sin x dx$. Since x is an algebraic function and $\sin x$ is trigonometric function, following LIATE, we choose $u = x, dv = \sin x dx$, which implying that $v = -\cos x$. Following integration by parts, we have $\int x \sin x = -x \cos x - \int -\cos x dx = -x \cos x + \sin x + c$. Similarly, to find $\int \ln x dx$, we choose $u = \ln x$, which is a logarithmic function, and $dv = dx$. Therefore we have $\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + c$.

D. Integration by substitution.

If the function is a function of	Substitute
$(a^2 - x^2)$ or $\sqrt{a^2 - x^2}$	$x = a \sin \theta$
$(x^2 - a^2)$ or $\sqrt{x^2 - a^2}$	$x = a \sec \theta$
$(x^2 + a^2)$ or $\sqrt{x^2 + a^2}$	$x = a \tan \theta$
$\int \frac{dx}{(ax+b)\sqrt{px+q}}$	$px+q = u^2$
$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$	$ax+b = \frac{1}{u}$

For example:

$$\begin{aligned}
 & \int \frac{dx}{x^2 \sqrt{x^2 - 16}} \\
 &= \int \frac{4 \sec \theta \tan \theta d\theta}{16 \sec^2 \theta \cdot 4 \tan \theta}, x = 4 \sec \theta, dx = 4 \sec \theta \tan \theta d\theta \\
 &= \frac{1}{16} \int \frac{d\theta}{\sec \theta} = \frac{1}{16} \int \cos \theta d\theta \\
 &= \frac{1}{16} \sin \theta + c = \frac{\sqrt{x^2 - 16}}{16x} + c
 \end{aligned}$$

E. Partial Fraction.

Evaluate the integral $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$.

Since the integrand is a (proper) rational function, the partial fraction method applies. For $x^3 + 4x = x(x^2 + 4)$, we can write $\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$.

Multiplying both sides by $x(x^2 + 4)$, we have

$$2x^2 - x + 4 = A(x^2 + 4) + Bx + C \quad x = (A + B)x^2 + Cx + 4A.$$

Comparing the coefficient, we obtain $A = 1, B = 1$ and $C = -1$, so

$$\begin{aligned}
 \int \frac{2x^2 - x + 4}{x^3 + 4x} dx &= \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4} \right) dx \\
 &= \int \frac{1}{x} dx + \int \frac{x}{x^2 + 4} dx - \int \frac{1}{x^2 + 4} dx \\
 &= \ln |x| + \frac{1}{2} \ln x^2 + 4 - \frac{1}{2} \tan^{-1} \frac{x}{2} + c
 \end{aligned}$$

4. **DEFINITE INTEGRAL:**

Consider a continuous function $f(x)$ a closed interval $[a, b]$, which is partitioned into

n sub-intervals of length $\Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_n$. Let x_i^* be any value of x on the i -th sub-interval. Then the definite integral $\int_a^b f(x)dx$ is defined as

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \cdot \Delta x_i.$$

Properties of definite integral.

- $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c \int_a^b f(x)dx.$
- $\int_a^a f(x)dx = 0.$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx.$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any c .

5. FUNDAMENTAL THEOREM OF CALCULUS:

If $f(x)$ is continuous function on a closed interval $[a, b]$, then

- $F(x) = \int_a^x f(t)dt$ is a continuous function on $[a, b]$, differentiable on (a, b) and $F'(x) = f(x)$ for any $x \in (a, b)$.
- $\int_a^b f(x)dx = F(x) \Big|_a^b = F(b) - F(a)$, where $F(x)$ is an antiderivative of $f(x)$.

Moreover, if $F(x) = \int_a^{g(x)} f(x)dx$, then $F'(x) = f(g(x)) \cdot g'(x)$.

For example, if $F(x) = \int_a^x t \sin t^3 dt$, then $F'(x) = x \sin x^3$, just replace the variable t in the integrand by the upper limit x . Moreover, if $F(x) = \int_a^{x^2} \sin t dt$, then $F'(x) = \sin x^2 \cdot 2x$.

6. IMPROPER INTEGRAL

A. Improper integral of type 1.

- If the integral $\int_a^t f(x)dx$ exists for any $t \geq a$, then $\int_a^\infty f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx$ if the limit exists.
- If the integral $\int_t^b f(x)dx$ exists for any $t \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx$ if the limit exists.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

- If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$.

For example, the improper integral $\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} (1 - \frac{1}{x^{p-1}})$ convergent if and only if $p > 1$. The improper integral $\int_{-\infty}^0 e^x dx = \lim_{t \rightarrow -\infty} \int_t^0 e^x dx = \lim_{t \rightarrow -\infty} (1 - e^t) = 1$, so $\int_{-\infty}^0 e^x dx$ is convergent.

B. Improper integral of type 2.

a. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b , the integral

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx \text{ provided that the limit exists.}$$

b. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a , the integral

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx \text{ provided that the limit exists.}$$

The improper integral $\int_a^b f(x) dx$ is convergent if the corresponding limit exists and divergent if the limit does not exist.

c. If $f(x)$ has discontinuity at c , where $a < c < b$, and both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ are convergent, then we define $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.

7. APPLICATION OF INTEGRALS

A. Arc-length. If $f'(x)$ is continuous on $[a, b]$, then the length of the curve

$$y = f(x), a \leq x \leq b \text{ is } L = \int_a^b \sqrt{1 + (f')^2} dx.$$

If a curve C is described by the parametric equation $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ where $f'(t)$ and $g'(t)$ are continuous on $[a, b]$ and C is traversed exactly once as t increases from a to b , then the length of C is $L = \int_a^b \sqrt{(f')^2 + (g')^2} dt$.

For example, the arc-length of the curve $y = 1 + 6x^{3/2}$ between $0 \leq x \leq 1$ is

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + 81x} dx = \frac{2}{243} (1 + 81x)^{3/2} \Big|_0^1 = \frac{2}{243} (82\sqrt{82} - 1).$$

B. Area between curves.

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ where f and g are continuous and $f \geq g$ for all x in $[a, b]$ is

$$A = \int_a^b (f(x) - g(x)) dx.$$

The area A of the region bounded by the curves $x = f(y)$, $x = g(y)$ and the lines $y = c$, $y = d$ where f and g are continuous and $f \geq g$ for all y in $[c, d]$ is

$$A = \int_c^d (f(y) - g(y)) dy.$$

For example, the area of the region bounded above by $y = e^x$ and $y = x$ and bounded on the sides by $x = 0$ and $x = 1$ is $\int_0^1 (e^x - x) dx = (e^x - \frac{x^2}{2}) \Big|_0^1 = e - \frac{3}{2}$.

C. Volume of solids of revolution.

Rotation about	x-axis	y-axis
Disk Formula	$V = \pi \int_a^b (f(x))^2 dx$	$V = \pi \int_c^d (f(y))^2 dy$
Washer Formula	$V = \pi \int_a^b (f(x) - g(x))^2 dx$	$V = \pi \int_c^d (f(y) - g(y))^2 dy$
Cylindrical Shell Formula	$V = \int_c^d 2\pi y f(y) dy$	$V = \int_a^b 2\pi x f(x) dx$
Difference of Shell Formula	$V = \int_c^d 2\pi y (f(y) - g(y)) dy$	$V = \int_a^b 2\pi x (f(x) - g(x)) dx$

Example: Let **R** be the region in the first quadrant bounded by the curves $y = x^3$ and $y = x^2 - 2x$. Calculate the following quantities:

- The volume obtained by rotating **R** about x -axis
- The volume obtained by rotating **R** about y -axis.

Solution: First, we need to find the region **R**. Solve $x^3 = x^2 - 2x$, and we have $x = 0, 1$ and 2 . The region **R**, which is in the first quadrant, is bounded below by $y = x^3$ and above by $y = x^2 - 2x$ with the endpoints $(0; 0)$ and $(1; 1)$.

By the washer formula, the volume of the solid obtained by rotating **R** about x -axis

$$\text{is } \int_0^1 \pi [(2x - x^2)^2 - (x^3)^2] dx = \frac{41}{105} \pi.$$

By Difference of Shells formula, the volume obtained by rotating **R** about y -axis is

$$\int_0^1 2\pi x [(2x - x^2) - (x^3)] dx = \frac{13}{30} \pi.$$

- Series. Consider a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$, let $s_n = a_1 + a_2 + \cdots + a_n$. If the sequence $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent, and we write $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = s$. The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.
 - Divergence test. If $\lim_{n \rightarrow \infty} a_n \neq 0$ or the limit does not exist, then $\sum a_n$ diverges.

For example, for $\lim_{n \rightarrow \infty} \cos \frac{1}{n} = 1 \neq 0$, it follows that $\sum \cos \frac{1}{n}$ is divergent.

B. Geometric series: A geometric series $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$ is convergent

if and only if $-1 < r < 1$. Moreover, its sum is $\frac{a}{1-r}$.

For example, the number $0.8888\dots$ can be written as a series

$0.8 + 0.08 + 0.008 + \dots$ with $a = 0.8$ and $r = 0.1$. The geometric series converges to

$$\frac{a}{1-r} = \frac{8}{9}, \text{ which implies that } 0.888\dots = \frac{8}{9}.$$

C. Positive series. The series $\sum a_n$ is a positive series if each term $a_n > 0$.

a. Integral test: Let $\sum a_n$ be a positive series and $f(x)$ be a continuous function such that $f(n) = a_n$. Then $\sum a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ is convergent.

The integral test implies that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

b. Comparison test: Let $\sum a_n$ and $\sum b_n$ are positive series and there exists an integer m such that $a_n < b_n$ for all $n > m$. Then

(1) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(2) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

For example, since $0 \leq \frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is convergent, it follows

that $\sum \frac{\cos^2 n}{n^2}$ is also convergent.

c. Limit comparison test. Let $\sum a_n$ and $\sum b_n$ are positive series. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l \neq 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

For example, since $\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$ and $\sum \frac{1}{n}$ is divergent, it follows that

$\sum \sin \frac{1}{n}$ is also divergent.

D. Alternating series test. An alternating series is a series whose terms are alternatively positive or negative.

If the alternating series $\sum_{n=1}^{\infty} (-1)^n b_n, (b_n > 0)$ satisfies

a. $b_n \geq b_{n+1}$ for all $n > 0$,

b. $\lim_{n \rightarrow \infty} b_n = 0$

Then the series $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

For example, the alternating series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ satisfies $\frac{1}{n} > \frac{1}{n+1}$ for any n and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

by alternating series test, we know the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is convergent.

E. General series. A series $\sum a_n$ is absolutely convergent if the series $\sum |a_n|$ is convergent. A series $\sum a_n$ is conditionally convergent if $\sum a_n$ is convergent but not absolutely convergent.

a. Ratio test. Let $\sum a_n$ be a series. Then

(1) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum a_n$ is absolutely convergent.

(2) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum a_n$ is divergent.

(3) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then no conclusion about convergence or divergence can be drawn.

For example, consider the series $\sum \frac{n^3}{3^n}$, since $\lim_{n \rightarrow \infty} \frac{(n+1)^3 / 3^{n+1}}{n^3 / 3^n} = \lim_{n \rightarrow \infty} \frac{(n+1)^3}{3n^3} = \frac{1}{3}$

, by the ratio test, we know $\sum \frac{n^3}{3^n}$ is convergent.

b. Root test: Let $\sum a_n$ be a series. Then

(1) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$, then the series $\sum a_n$ is absolute convergent.

(2) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$, then the series $\sum a_n$ is divergent.

(3) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, then no conclusion about convergence or divergence can be drawn.

For example, consider the series $\sum \left(\frac{2n+1}{3n-1}\right)^n$, since $\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{2n+1}{3n-1}\right)^n} = \lim_{n \rightarrow \infty} \frac{2n+1}{3n-1} = \frac{2}{3}$, by

root test, we know the series $\sum \left(\frac{2n+1}{3n-1}\right)^n$ is convergent.

9. Power series. A power series is a series of the form

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$ where x is a variable and c_n 's are constant, called coefficients of the series.

More generally, a series of the form

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$ is called *a power series in a* or *a power series centered at a* or *a power series about a* .

If a function $f(x)$ has a power series representation at a , that is, if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$,

then its coefficients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$.

Some functions:

A. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$.

B. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

C. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

D. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

E. $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

F. $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$.
