Tsiolkovsky Rocket Equation

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Abstract

In this project we derive the Tsiolkovsky rocket equation applying the momentum conservation principle to a system of varying masses, considering the gravitational field acting upon the rocket as it leaves a planet. Lastly, I apply the equations to calculate the final hight of a Saturn V rocket as a function of its change in mass.

1 Introduction

In 1813, the British mathematician William Moore drafted for the first time an equation that described the motion of rocket-like objects in space. This equation was of critical importance for humanity to understand how an object could travel in a frame in which no air can be pushed back. The model based on the third law of Newton, which describes the conservation of momentum of a system of objects pushing each other away, so that scientists and engineers could build engines that move rockets by thrusting part of the mass thereof. William Moore succeeded in relating the change in mass of the rocket and its momentum, and the mass of propellant expelled; however, he failed to account for the velocity at which the mass could be propelled: the exhaust velocity. Even though it was the Scottish minister William Leitch who derived the ideal rocket equation including the exhaust velocity, we honor Konstantin Tsiolkovsky for being the first one to apply the ideal rocket equation to real calculations of how much mass a rocket needs to carry and expel at a certain velocity to transfer momentum to the rocket and attain velocities necessary for space travel. However, when a rocket is either being launched or descending to a planet with an atmosphere, the effects of gravitational forces need be accounted for in a different model.

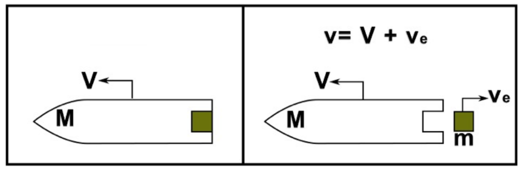

Figure 1: A rocket expels part of its mass to propel itself, seen from a laboratory frame.

2 Theory Outlines

A rocket traveling in space thrusts part of its mass to propel itself, with a constant force and in the opposite direction of its movement. According to the momentum conservation principle, the momentum lost by one object is gained by another object

$$\Delta(MV) = \Delta(mv), \tag{1}$$

where M and V are the mass and velocity of the rocket, and m and v are the mass and speed of the propellant. The motion of this rocket can be described from a laboratory frame as in Figure 1, from which

$$v = V + v_e, \tag{2}$$

where dm is the mass of propellant, v is the relative velocity of dm from a laboratory frame, and v_e is the exhaust velocity, as seen in $Figure 1$. The mass of the rocket (M) changes in time and equals the change in the mass expelled. Note that the change in mass is negative as the mass of the rocket decreases

$$dM = -dm, \tag{3}$$

where dM is the change of the mass of the rocket. Because momentum is conserved, the change in momentum of the rocket plus the change in momentum of the propelled mass equals zero

$$d(MV) + dm(V + v_e) = 0. \tag{4}$$

Replacing equation 2 in equation 3, and using the product rule we obtain:

$$MdV + v_e dM = 0, \tag{5}$$

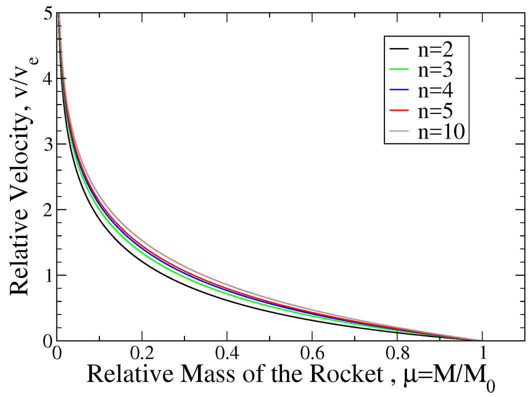

Figure 2: The velocity of the rocket increases as the ratio of its mass decreases from 1 to 0. The final velocity of the rocket does not change significantly for different n values.

where the mass of the rocket M times the change in velocity of the rocket equals the exhaust velocity v_e, which is opposite in direction to V, multiplied by the change in mass of propellant. Equation (5) can be integrated with respect to the change in mass of the rocket, from initial to final mass as

$$\int_{V_0}^{V} dV = -v_e \int_{M_0}^{M} \frac{dM}{M}, \tag{6}$$

which gives the change in velocity as a function of the change in mass,

$$\Delta V = -v_e \ln\left(\frac{M}{M_0}\right). \tag{7}$$

Solving for mass as a function of change in velocity we get:

$$M = M_0 e^{-\frac{\Delta V}{v_e}}. \tag{8}$$

We need to account for the velocity the rocket loses as it leaves a gravitational field. Thus, $-gt$ is included in the equation.

$$\Delta V = -v_e \ln\left(\frac{M}{M_o}\right) - gt, \tag{9}$$

Where t is the time at which all propellant is used. Assuming that the rate of fuel consumption does not change, the initial mass a rocket needs in order to attain certain velocity as it escapes gravity is thus given by:

$$M = M_0 e^{\frac{-gt-\Delta V}{v_e}}. \tag{10}$$

Let us consider a rocket moving at constant thrust $T = -v_e\dot{M} = constant$ where $\dot{M}$ is the gas flow rate of the rocket, which is launched vertically upwards from rest. If $\dot{M} = constant$ then,

$$M = M_0 + |\dot{M}|t, \tag{11}$$

is a linear function of time. Considering the gravity force and T, the velocity is given by

$$v = v_0 - v_e \ln(M/M_0) - gt = v_0 - v_e \ln(M/M_0) - g(M - M_0)/\dot{M}. \tag{12}$$

We take $\mu = M/M_0$ as a new variable, and $n = -v_e\dot{M}/(M_0 g)$ as a ratio between the thrust of a rocket and its initial weight, which is positive and constant. Additionally, if $V_0 = 0$, then,

$$v = -v_e \ln(\mu) + v_e(\mu - 1)/n, \tag{13}$$

where $n > 1$ and $\mu <= 1$. We can integrate equation (14) over time to find the final height of the rocket as

$$h = \int V dt = \int v_e(-\ln(\mu) - \frac{(1-\mu)}{\mu})dt. \tag{14}$$

We then solve equation (11) for the change in time $dt = -\frac{M_0}{|\dot{M}|}d\mu$ and substitute in equation (14) to obtain:

$$h = \frac{v_e^2}{gn}\int_\mu^1 (-\ln(\mu) - \frac{(1-\mu)}{n})d\mu. \tag{15}$$

Which yields,

$$h = \frac{v_e^2}{gn}(\mu \ln(\mu) - \mu + 1 - \frac{(\mu-1)^2}{2n}) \tag{16}$$

3 Results

Let us take the Saturn V rocket and calculate the final height of the rocket for different $\mu's$. The mass of the Saturn V is $M = 2,970,000kg$, its exhaust velocity for LOX/LH2 is $v_e = 4,400ms^{-1}$, and its thrust is $T = 9,200,000lb = 40,923,639N$. When the rocket has used half of its mass $\mu = 0.5$, its final height is given by:

$$h = \frac{(4,400\frac{m}{s})^2}{9.81\frac{m}{s^2} \cdot \frac{40,923,639N}{2,970,000kg \cdot 9.81\frac{m}{s^2}}} \cdot (0.5\ln(0.5) - 0.5 + 1 - \frac{(0.5-1)^2}{2 \cdot \frac{40,923,639N}{2,970,000kg \cdot 9.81\frac{m}{s^2}}}) = 90.5km. \tag{17}$$

The Saturn V rocket can reach a height of 90.5km above Earth's surface if it uses half of its mass. Different heights for different $\mu's$ are given in the table

h=h(μ)	
μ	$h_{final}(km)$
0.1	535.9
0.2	351.7
0.3	231.0
0.4	148.0
0.5	90.5
0.6	51.4
0.7	25.7
0.8	10.2
0.9	2.3
1	0

In order to go to the International Space Station, whose orbit height is 408km, a Saturn V rocket would use from 80 to 90 percent of its mass. Furthermore, in the Apollo 11 mission, the engines of the rocket were shut down at a height of 334km, at which the gravity pull from Earth was negligible, and thus the rocket would keep traveling in space. It is interesting that most of the mass of the rocket is used to escape earth.