

Beta Equilibrium in Neutron Stars

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Abstract

A free neutron is an unstable particle with a mean lifetime of about 15 minutes, meaning that after this time it most likely will decay. Why do the neutrons not decay in a neutron star which is composed of neutrons? In this research we study the beta-equilibrium in neutron stars, which prevents the neutron from decaying and saves the star. The equilibrium is reached when the beta decay process (when a neutron decays into proton and an electron) is balanced out by the inverse beta decay (when a proton and an electron produce a neutron). To find the beta equilibrium we treated a neutron star as a Fermi Gas of free neutrons, protons, and electrons at absolute zero temperature. We found that in order to be stable a neutron star, must have certain amount of protons and electrons, with the ratio of number of protons to the number of neutrons is about 0.0012%.

1 Introduction

The neutron stars (NS) have been predicted and measured by many scientists since the mid-20 century. The understanding of the formation and properties of NS is very important for modern astronomy. In general, NS is calculated to have mostly 1.6-2.1 solar mass with the radius of around 10-22 kilometers. Every NS consists of mostly neutrons and a small number of protons and electrons. It is an important fact that neutrons are not stable and decay in free space in about 15 minutes. But why do we have stable neutrons inside a NS and inside its nuclei? The main goal of this research is to answer this question by analyzing the process of beta decay and its equilibrium inside a NS.

Understanding the mechanism of beta decay inside NS is essential to grasp how NS operate in space. Beta decay is the process when a neutron is converted into a proton and an electron (with neutrinos and energy which will be discussed later in discussion). Neutrons are not stable in free space and always have a tendency to decay into protons and electrons. They are stable in the nuclei of atoms because of the nuclear force. The nuclear force is super strong but only exist in about 10^{-15} meter. NS have a density similar to the nuclei. This density is formed by the gravity force rather than the nuclear force. The beta decay stops and reaches the equilibrium when the energy of neutron is less than the energy of protons and electrons.

We use the ideal Fermi-gas model to discuss the energy of fermions. In the Fermi-gas model, every momentum phase can only be occupied by one identical fermion. In our case, electrons and protons and neutrons are different fermions and can exist in the same space. The beta decay still exists in the NS due to factors that affect the ideal case like the temperature, but it is proved to reach an equilibrium theoretically when the energy of protons and electrons get too high for neutrons to decay into.

2 Theory

2.1 Fermi-Gas model and Momentum space

In NS, we take neutrons as Fermi-gas model because neutrons are not reacting together. In Fermi-gas model, there exists a smallest momentum space for one particle to take up. In a 1-dimension (1-D) phase space, the minimum phase space of one particle is $2\pi\hbar$ base on the uncertainty principle. Every minimum phase space can only be occupied by one identical particle. Different particles are independent in phase space and they can occupy the same phase space at the same time.

Fermions always take up the lower energy space first and then move up(from yellow to red):

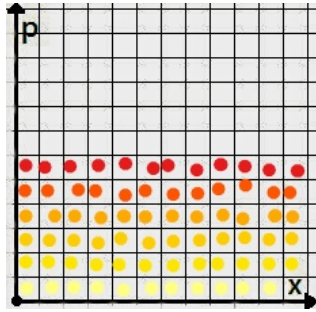


Figure 1: This graph shows a 2-dimension (2-D) space consists of x-axis and p-axis. $p = mv$ is the momentum of each particle, x is the size of the space. Each box is the minimum 1-D phase space ($2\pi\hbar$) that a fermion can take up. From yellow to red, the energy of the particles increases.

The number of particles N in 1-D phase space can be calculated with:

$$N = 2 \frac{p x}{(2\pi\hbar)}, \quad (1)$$

where $\hbar = 1.055 \times 10^{-34} Js$ is the reduced Planck constant, and the coefficient 2 is due to the neutron spin 1/2.

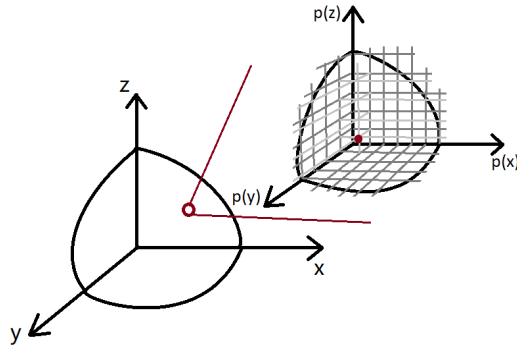


Figure 2: This graph shows a 6-dimension (6-D) space when 3-dimension (3-D) momentum space is composed with 3-D real space. Particles of different momentum can exist in the same point in 3-dimension space.

We take the 6-D space as 3 1-dimensional phase space with each phase space as a 2-D space as Figure 1. Take the space as a sphere, the total phase

space (ps) in 3-dimension is:

$$V_{sphere} = \frac{4\pi}{3}R^3 \rightarrow V_{ps} = \frac{4\pi R_n^3}{3} \cdot \frac{4\pi p_{max}^3}{3}, \quad (2)$$

where R_n is the radius of NS.

The minimum 6-D space (V_{min}) can be treated as the multiplication of 3 1-D phase space which is a 2-D space with one p-axis and one x-axis:

$$V_{min} = (2\pi\hbar)^3 \quad (3)$$

where $2\pi\hbar$ the minimum space in 1-D phase space.

The number of particles N in 6-D space, or 3-D phase space can be calculated with:

$$N = \frac{2}{(2\pi\hbar)^3} \frac{4\pi R_n^3}{3} \cdot \frac{4\pi p_{max}^3}{3}. \quad (4)$$

2.2 The energy balance and the equilibrium state

For the neutrons in the neutron star, we can introduce their maximum energy:

$$E_{max}(\text{neutron}) = \frac{p_{max}^2}{2m_n}, \quad (5)$$

On the same state of momentum, the relation between p and the energy of particles will be:

$$\left. \begin{array}{l} P = mv \\ E = \frac{mv^2}{2} \end{array} \right\} \rightarrow E = \frac{pv}{2} = \frac{p^2}{2m}, \quad (6)$$

$$\rightarrow E \propto p^2, E \propto 1/m.$$

To obey the Fermi-Dirac statistic, the momentum of identical particle will increase as the number of particles increase. If NS starts with only neutrons, the highest energy of neutrons will lead to the beta decay reaction because particles always prefer to stay in a lower energy state. When the high-energy neutrons become protons and electrons, they can occupy lower momentum space without breaking the the Fermi-Dirac statistic and reach a lower energy state.

m_e is much less than m_n and m_p , according to the equation (6), energy of electrons will rise much faster than neutron and proton when they all occupy the same phase space:

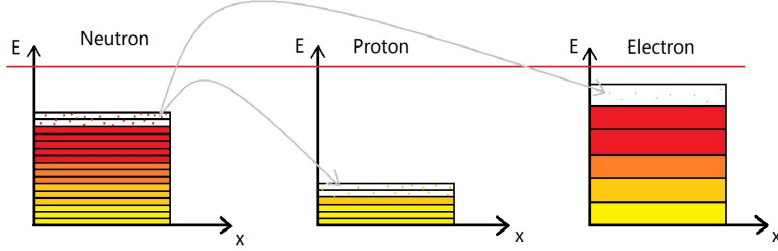


Figure 3: Fermi energy diagram for the neutrons (the left panel), the protons (the middle panel), and the electrons (the right panel). As the graph shows, the highest energy level for every type of Fermion is the surface energy (E_s). Inverse beta decay starts when the E_s of electrons and protons is higher than the E_s of neutrons, which is what the diagram shows.

Beta decay and inverse beta decay is balanced out when the E_s of electrons and protons is equal to the E_s of neutrons

Highest energy, E_s of the three particles are:

$$\begin{aligned} E_s(\text{neutron}) &= \frac{p_n^2}{2m_n}, \\ E_s(\text{proton}) &= \frac{p_p^2}{2m_p}, \\ E_s(\text{electron}) &= \frac{p_e^2}{2m_e}. \end{aligned} \quad (7)$$

Base on the law of conservation of energy, we reach equilibrium state when the $E_s(\text{neutron})$ is about equal to the sum of $E_s(\text{proton})$ and $E_s(\text{electron})$. The equilibrium state can be described with:

$$E_{max}(\text{neutron}) \leq E_{max}(\text{proton}) + E_{max}(\text{electron}). \quad (8)$$

And a ideal equilibrium state can be described with:

$$E_{max}(\text{neutron}) = E_{max}(\text{proton}) + E_{max}(\text{electron}). \quad (9)$$

2.3 Result

If the neutron star has only neutrons, the number of neutrons is

$$N = M_{NS}/M_n \quad (10)$$

According to equation (4), maximum momentum of neutrons is:

$$N = \frac{2}{(2\pi\hbar)^3} \cdot \frac{4\pi R^3}{3} \cdot \frac{4\pi p_{max}^3}{3} \rightarrow p_{max} = \sqrt[3]{\frac{9N(2\pi\hbar)^3}{16\pi^2 R^3}} = \sqrt[3]{\frac{9\pi N\hbar^3}{2R^3}}. \quad (11)$$

The ratio of mass of proton to the mass of electron is about 10^6 . According to equation (7), $E_s(\text{proton}) = 10^{-6} \cdot E_s(\text{electron})$. So the $E_s(\text{proton})$ is negligible.

And the equilibrium state is reached when $E_s(\text{neutron}) = E_s(\text{electron})$:

$$\left. \begin{aligned} E_s(\text{neutron}) &= \frac{p_n^2}{2m_n}, \\ E_s(\text{electron}) &= \frac{p_e^2}{2m_e} \\ E_s(\text{neutron}) &= E_s(\text{electron}). \end{aligned} \right\} \rightarrow \frac{p_{max}^2(\text{neutron})}{2m_n} = \frac{p_{max}^2(\text{electron})}{2m_e} \quad (12)$$

In the same NS, three different particles exist in the same space with the same radius $R = R_{NS}$. Based on the equation (11) and (12):

$$\left(\frac{9\pi N_n \hbar^3}{2R^3}\right)^{2/3} / 2m_n = \left(\frac{9\pi N_e \hbar^3}{2R^3}\right)^{2/3} / 2m_e, \quad (13)$$

where the N_n is the number of neutrons and N_e is the number of electrons, which is the same as N_p .

The ratio between the number of neutrons and the number of protons or electrons can be calculated with:

$$\begin{aligned} \frac{2m_n}{2m_e} &= \frac{\left(\frac{9\pi N_n \hbar^3}{2R^3}\right)^{2/3}}{\left(\frac{9\pi N_e \hbar^3}{2R^3}\right)^{2/3}} \\ \rightarrow \frac{m_n}{m_e} &= \left(\frac{N_n}{N_e}\right)^{2/3} \end{aligned} \quad (14)$$

$$\rightarrow \frac{N_{\text{electron}}}{N_{\text{neutron}}} = \left(\frac{m_{\text{electron}}}{m_{\text{neutron}}}\right)^{3/2} \approx \left(\frac{1}{2000}\right)^{3/2} = 0.000011180334 \approx 0.000012$$

3 Discussion

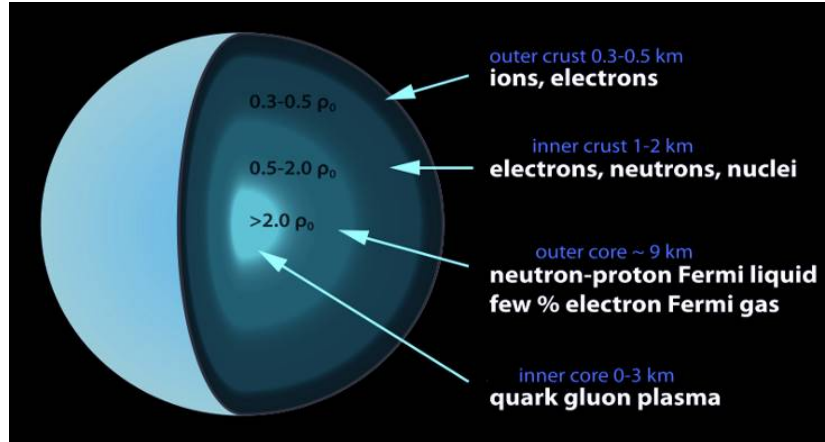


Figure 4: Interior of a neutron star. The picture is taken from [3].

1.) The neutrino carries out the energy and the NS cool down due to neutrino. Would the energy neutrino bring away affect the equilibrium? The ratio of the energy that neutrinos take away to the whole energy of each neutron or proton is actually so small that it is negligible in each reaction. The NS cool down because there are billions of neutrinos flying away every second.

2.) Why we can take NS as a fermi-gas model? NS is discovered to have four layers: inner core (0-3 km), outer core (about 9 km), inner crust (1-2 km) and outer crust (0.3-0.5 km). The inner core is too dense to describe (called the nuclear pasta). The outer core exists as liquid and fermi gas. We study the whole neutron star as fermi gas ignoring the crust and inner core since outer core takes up the biggest percentage of the NS.

Even though the ideal fermi-gas model should be at absolute 0 temperature and the NS is hot, the energy of particles is relatively high, so we can almost ignore the effect of temperature. When NS cooled down, the temperature of NS is about 106 Kelvin. And the maximum energy of neutrons is about 0.1 GeV, which is 10^{12} K. Temperature of the NS may cause some fermions jumping from lower energy state to higher energy state and enable particles to continue the reaction after reaching the equilibrium.

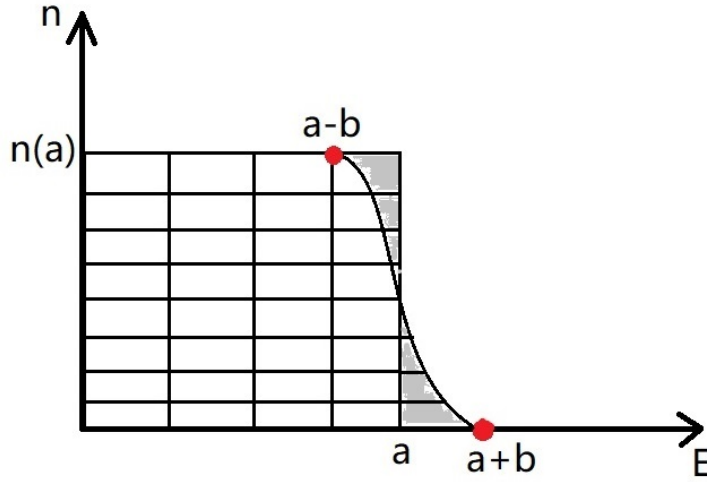


Figure 5: n is the number of neutrons in each energy state, and E is the energy state of neutrons. a is the highest energy that a neutron can have, which is 10^{12} K in this case, and b is the highest energy that the temperature of NS can provide, which is 10^6 K here.

4 Conclusion

Ignoring the minor effects in our discussion, beta equilibrium in NS of any size is reached when the ratio of neutrons to protons is approximately $250000/3$.